Computational Game Theory	Fall Semester, 2012

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Lecturer: Amos Fiat *Scribe: Alex Fonar, Eyal Dushkin and Slava Novgorodov*

13.1 Single Minded Bidders

13.1.1 Introduction

This lecture focuses on the twin goals of computational complexity and strategic behavior, through observation on a "single-minded bidders" valuations. Such bidders are interested only in a single sepcified bundle of items, and get a sepcified scalar value if they get this whole bundle (or any superset) and get zero value for any other bundle[1]. In general, maxmizing revenue in a "single-minded bidders" valuations is as hard as "weighted-packing" problem which is NP-Complete followed by reduction from the INDEPENDENT-SET problem. We'll present 3 polynomial algorithms, produce an approximiation to the optimal welfare.

Definition A valuation v is called **single minded** if there exists a bundle of items S^* and a value $v^* \in \Re^+$ such that $v(S) = v^*$ for all $S \supseteq S^*$, and $v(S) = 0$ for all other S. A single-minded bid is the pair (S^*, v^*).

Given n players, m items and single-minded pairs (S_i,v_i) $\forall i \in \{1, .., n\}$, we would like to maximize the social welfare in meaning of maximize revenue from selling items. Formally we would like to find a subset of winning bids $W \subseteq \{1, ..., n\}$ such that for every $i \neq j \in W, S_i \cap S_j = \emptyset$ with maximum social welfare $\sum v_i, i \in W$.

13.1.2 Greedy algorithm(sorted by v_i)

Sort the players descendingly by their prices. Scan the sorted list, and add player's single-minded pair to the solution, if possible (i.e. if the player is

[1] Algorithmic Game Theory - Nisan, Rounghgarden, Tardos and Vazirani. p.270

compatible with the other players in the solution). This algorithm gives only $\Omega(m)$ approximation to optimal solution.

Proof Consider an auction with m items and $m+1$ players. $\forall i \in \{1,..,n\}$, player i proposes 1 dollar for the singleton bundle contains item i, $\{i\}$. Player $m+1$ proposes $1+\epsilon$ for the bundle contains all items, $\{1,...,m\}$. In this case the above greedy algorithm outputs a single winner with a revenue of $1+\epsilon$, while the optimal solution stands on a revenue of m (by taking the offers of first m players).

13.1.3 Greedy algorithm(sorted by $v_i/|S_i|$)

Sort the players descendingly by $v_i/|S_i|$. Scan the sorted list, and add player's single-minded pair to the solution, if possible. This algorithm also gives only $\Omega(m)$ approximation to optimal solution.

Proof Consider an auction with m items and 2 players. Player 1 proposes $1+\epsilon$ for the first item. Player 2 proposes m for all items. In this case the algorithm outputs a revnue of $1+\epsilon$, while the optimal solution stands on a revnue of m(simply by taking the offer of the second player).

13.1.4 Greedy algorithm(sorted by $v_i/\sqrt{|S_i|}$)

Sort the players descendingly by $v_i/\sqrt{|S_i|}$. Scan the sorted list, and add player's single-minded pair to the solution, if possible. In this case, the algorithm produces a $\theta(\sqrt{m})$-aapproximation to the optimal solution. Moreover, approximating the optimal allocation among single-minded bidders to within a factor better than $m^{1/2\text{-}\epsilon}$ is NP-hard[1].

The Greedy Mechanism for Single-Minded Bidders:

Initialization:

- Reorder the bids such that $v_1^*/\sqrt{|S_1^*|} \geq v_2^*/\sqrt{|S_2^*|} \geq ... \geq v_n^*/\sqrt{|S_n^*|}$.
- $W \leftarrow \emptyset$.

For i = 1...n do: if $S_i^* \cap \left(\bigcup_{j\in W} S_j^*\right) = \emptyset$ then $W \leftarrow W \cup \{i\}$.

Output:

Allocation: The set of winners is W.

Payments: For each $i \in W$, $p_i = v_j^*/\sqrt{|S_j^*|/|S_i^*|}$, where j is the smallest index such that $S_i^* \cap S_j^* \neq \emptyset$, and for all $k < j, k \neq i$, $S_k^* \cap S_j^* = \emptyset$ (if no such j exists then $p_i = 0$).

[1] Algorithmic Game Theory - Nisan, Rounghgarden, Tardos and Vazirani. p.272

Theorem 13.1 The greedy algorithm(sorted by $v_i/\sqrt{|S_i|}$) produces a $\theta(\sqrt{m})$-approximation of the optimal social welfare.

Proof Lets denote group of players served, by I. Group of players served in optimal solution wiil be I*. Lets sort all players by $v_i/\sqrt{|S_i|}$) from big to low. We part I* in following way:

i ∈ I*, i ∈ I ⇒ i is in group of i (so i has group named by his name)

i ∈ I*, i ∉ I ⇒ i is in group of j (when j is player with his own group j, and so j ∈ I)

Lets denote group of players that we associate with i by F_i.

So $\{F_i\}$ is partition of I*.

$F_i \bigcap F_j = \emptyset$ if $i \neq j$

Maximal social wellfare = OPT = $\sum_{i*\in I*} v_i* = \sum_{i\in I} \sum_{i*\in F_i} v_i*$ (1)

Now by alghorithm:

$\frac{v_i*}{\sqrt{|S_i*|}} \leq \frac{v_i}{\sqrt{|S_i|}}$

and then:

$v_i* \leq \frac{v_i\sqrt{|S_i*|}}{\sqrt{|S_i|}}$

So:

$(1) \leq \sum_{i\in I} \frac{v_i}{\sqrt{|S_i|)}} \sum_{i*\in F_i} \sqrt{|S_i|} \leq \sum_{i\in I} \frac{v_i}{\sqrt{|S_i|}} \sum_{i*\in F_i} \sqrt{\frac{m}{|F_i|}} * |F_i|(2)$

This because $m \geq \sum_{i*\in F_i} |S_i *|$ and square root is concave function so $\sum_{i=1}^{n} \sqrt{a_i} \leq n\sqrt{\frac{\sum_{i=1}^{n} a_i}{n}}$ for every positive numbers a_i

And now last accord:

$(2) = \sum_{i\in I} \frac{v_i}{\sqrt{|S_i|}} \sqrt{m|F_i|} = \sqrt{m} \sum_{i\in I} v_i \sqrt{\frac{|F_i|}{|S_i|}} \leq \sqrt{m} \sum_{i\in I} v_i =$

$= \sqrt{m}$ * Approximated social wellfare