4-node clique network bargaining solition

Alex Fonar
I.D. 323301515

1 Introduction

As you asked, I send you full (well, almost full) bargaining solution in case of 4-node clique network(Fig 1).

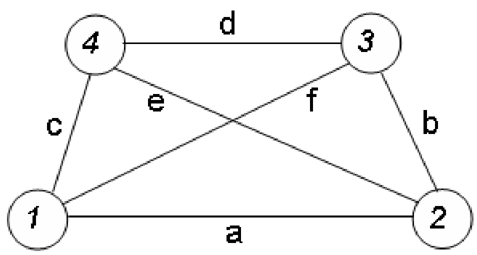

Figure 1: 4 - node network

Kleinberg and Tardos proved that in case balanced outcome exist, balanced outcome agreements will be reached on the edges of maximal matching. I assume that maximal matching consists of edges a and d, and also $a \geq d$. Moreover, if there is some stable outcome, then balanced outcome exist too. So, it's sufficient to find stable outcome for proving balanced outcome existence. Now, we define $t = max\{c, f\}, s = max\{e, b\}$. There are 2 cases: t and s form matching (if $(t, s) = (c, b)$ or $(t, s) = (f, e)$), or not (if $(t, s) = (f, b)$ or $(t, s) = (c, e)$).

2 t and s doesn't form matching

More interesting case. Say, $t = f$ and $s = b$ (Fig. 2).

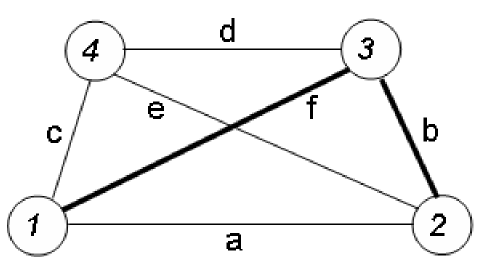

Figure 2

So, $f \geq c$ and $b \geq e$. And $\{a, d\}$ is maximal matching, so $a + d \geq e + f, b + c$. Solution depends on value of $\frac{f+b-a}{2}$.

2.1 $\frac{f+b-a}{2} \leq \frac{d}{2}$

Or simply $f + b - a \leq d$.
There is stable outcome:
$(\frac{a+f-b}{2}, \frac{a+b-f}{2}, \frac{d}{2}, \frac{d}{2})$

Stability can be proved easily.

2.2 $\frac{d}{2} \leq \frac{f+b-a}{2} \leq d$

There is stable outcome:
$(\frac{a+f-b}{2}, \frac{a+b-f}{2}, \frac{f+b-a}{2}, d - \frac{f+b-a}{2})$

Here, I will prove stability only for node 4. For other case it is quite similar.
Stability inequalities for node 4 are: $c - \frac{a+f-b}{2} \leq d - \frac{f+b-a}{2}$ and $e - \frac{a+b-f}{2} \leq d - \frac{f+b-a}{2}$. They are identical to $c \leq d + a - b$ and $e \leq d + a - f$, which are true, because $\{a, d\}$ is maximal matching.

2.3 $\frac{f+b-a}{2} > d$

if $d < \frac{f+b-a}{2}$, then also $c < \frac{a+f-b}{2}$ and $e < \frac{a+b-f}{2}$.
($\frac{a+f-b}{2} = \frac{a+f+b}{2} - b = \frac{f+b-a}{2} + a - b > d + a - b \geq c$. For e prove is quite same.)
In this case, we have standard *vulnerability* case for triangle {1,2,3}, when node 4 is out of game. *Vulnerability* solution states, that nodes (1,2,3) has assigned values of $(\frac{a+f-b}{2}, \frac{a+b-f}{2}, \frac{f+b-a}{2})$, when only 2 nodes out of 3 get assigned amount. Node 4 cannot interfere, because assigned amounts are greater than sums node 4 can propose. In the end, two nodes out of {1,2,3} will reach an agreement and receive assigned amounts of money, and third node will reach an agreement with node 4 on equal terms. So there is 3 posiibillities:

- Nodes 1 and 2 reach an agreement on edge a. Allocation will be $(\frac{a+f-b}{2}, \frac{a+b-f}{2}, \frac{d}{2}, \frac{d}{2})$.
- Nodes 2 and 3 reach an agreement on edge b. Allocation will be $(\frac{c}{2}, \frac{a+b-f}{2}, \frac{f+b-a}{2}, \frac{c}{2})$.
- Nodes 1 and 3 reach an agreement on edge f. Allocation will be $(\frac{a+f-b}{2}, \frac{e}{2}, \frac{f+b-a}{2}, \frac{e}{2})$.

3 t and s form matching

Say, $t = c$ and $s = b$ (Fig. 3). So $c \geq f$ and $b \geq e$. Still, I assume that $\{a, d\}$ is maximal matching and $a \geq d$.

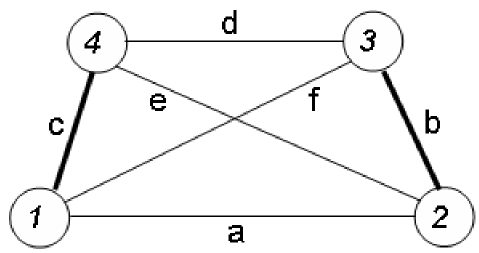

Figure 3

It seems that in this situation balanced outcome always exist. I'm not sure, cause I didn't find solution for one rare case.

3.1 $c, b \leq \frac{d}{2}$ or $c, b \geq \frac{d}{2}$

There is stable outcome:
$(\frac{a+c-b}{2}, \frac{a+b-c}{2}, \frac{d}{2}, \frac{d}{2})$
In case of $c, b \geq \frac{d}{2}$, it is also balanced.

3.2 $c < \frac{d}{2}$ and $b > \frac{d}{2}$ (symmetrical case is exactly the same)

There is stable outcome: $(\gamma_1, a-\gamma_1, d-\gamma_4, \gamma_4)$, where γ_1 and γ_4 hold following inequalities:
$c \leq \gamma_4 + \gamma_1 \leq (a + d) - b$
$e - a \leq \gamma_4 - \gamma_1 \leq d - f$
These inequlities always have solution, but we still need it to be not negative. I believe, that there is always values that meets all the conditions. There are some subcases:

- If $e \leq a + c$, there is stable solution $(0, a, d - c, c)$
- If $e > a + c$ and $b + e \leq 2a + d$, stable solution is $(\frac{2a+d-b-e}{2}, \frac{b+e-d}{2}, \frac{d+b-e}{2}, \frac{d+e-b}{2})$

I don't have solution to case where $c < \frac{d}{2}, b > \frac{d}{2}, e > a + c$ and $b + e > 2a + d$.