Computational Game Theory	Spring Semester, 2011/2012
Lecture 5: April 18	
Lecturer: Amos Fiat	*Scribe: Miki Shifman & Eldad Rubinstein*

5.1 Network Creation Game with Arbitrary Cost Partition

$G = (V, E)$ is an undirected graph with arbitrary edge costs $c(e)$. There are k players. Each player i has a source s_i and a sink t_i he wants to have connected.

Player i picks payment $p_i(e)$ for each edge e. e is *bought* if total payments $\geq c(e)$. When an edge is bought, all players can use it.

Each player i has only two concerns:

1. Must be a bought path from s_i to t_i
2. Given this requirement, i wants to pays as little as possible.

Player i doesn't care whether other players connect.

We will show that there is no Pure Nash Equilibrium in this game, and therefore it cannot be a Congestion Game.

5.1.1 3 Observations

1. The bought edges in a NE form a forest. (Otherwise, there are edges that can be renounced.)
2. Players only contribute to edges on their s_i-t_i path in this forest.
3. The total payment for any edge e is either $c(e)$ or 0.

5.1.2 Example - No Pure NE

There are two players, and all edges cost 1 (See Figure 5.1 on page 2). We know that any NE must be a tree: WLOG assume the tree is a, b, c.

- Only player 1 can contribute to a.
- Only player 2 can contribute to c.

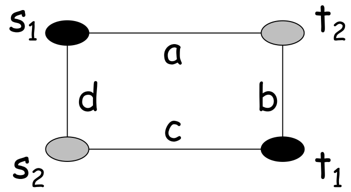

Figure 5.1: Network Creation Example, no Pure NE

- If player 1 or player 2 contributes to b, the situation becomes unstable. (e.g., player 2 would prefer paying 1 for edge d and using edge a for free than paying $1+\varepsilon$ for edge c and part of edge b.)

As we can see, there is no Pure NE in this example.

5.2 Smooth Games and Intrinsic Robustness

5.2.1 Atomic identical flow

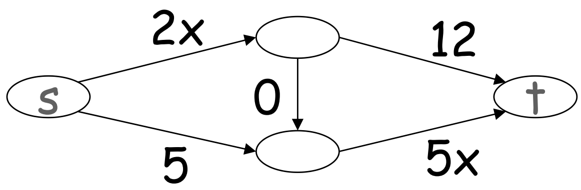

Figure 5.2: an example flow

Identical flow is a flow where every player has a single “car” that he would like to drive from s to t.

The flow over edge may be atomic such that a few players can use the same edge concurrently. the cost of each edge (resouce) is multiplied by the number of players using it concurrently.

5.2.2 Costs of atomic and non-atomic flow

Definition The cost $C(f)$ *of flow* f *is the sum of all costs incurred by traffic (avg cost* $\times$ *traffic rate)*

$$C(f) = \sum_p f_p c_p(f)$$

Theorem 5.1 *Atomic idential flow is a congestion game.*

5.2.3 Linear costs

Definition linear cost $f(n)$ is in the form of $c_e(x) = a_e(x) + b_e$ where e is an edge and x is the flow in edge e.

We will discuss cases of an atomic flow where every edge has the same cost of 1. Therefore, x is the number of players using the edge e.

Figure 5.2 is an example for the difference between NE and optimal cost.

5.2.4 POA for non-atomic flow

Theorem 5.2 *For every non-atomic flow network with linear costs, the cost of non-atomic Nash flow* $\leq \frac{4}{3}\times$ *the cost of optimal flow.*

5.2.5 Smoothness and its relation to POA

We will define the following abstract setup:

- n players, the strategy for player i is s_i
- Player i incurs cost $C_i(\mathbf{s})$
- Objective function is social welfare: min $\text{cost}(\mathbf{s}) := \sum_i C_i(\mathbf{s})$

Definition A game is (λ, μ)-*smooth* if for every pair $\mathbf{s}, \mathbf{s}^*$ outcomes (for $\lambda > 0$, $\mu < 1$):

$$\sum_i C_i\left(s_i, s_i^*\right) \leq \lambda{\cdot}\text{cost}(\mathbf{s}^*) + \mu \cdot \text{cost}(\mathbf{s})$$

Using smoothness, we will upper bound POA.

$$\text{cost}(\mathbf{s}) =_1 \sum_i C_i(\mathbf{s}) \leq_2 \sum_i C_i\left(s_i^*, s_{-i}\right) \leq_3 \lambda \cdot \text{cost}(\mathbf{s}^*) + \mu \cdot \text{cost}(\mathbf{s})$$

Transition 1 is by definition of cost, transition 2 is according to NE, transition 3 is using smoothness argument.

Therefore, POA (of pure NE) $\leq \frac{\lambda}{1-\mu}$. The best (λ, μ)-smoothness parameters are those minimizing $\frac{\lambda}{1-\mu}$.

Lemma 5.3 *For all non-negative integers y, z:* $y\left(z+1\right) = \frac{5}{3}y^2 + \frac{1}{3}z^2$.

We will use the lemma to prove an upper bound on POA.

For $a, b \geq 0$: $y(z+1) = \frac{5}{3}y^2 + \frac{1}{3}z^2$ and therfore: $ay(z+1) + by = \frac{5}{3}\left(ay^2 + by\right) + \frac{1}{3}\left(az^2 + bz\right)$

Let $\mathbf{s}, \mathbf{s}^*$ be any two vectors of strategies in a congestion game, with loads $\mathbf{x}$ and $\mathbf{x}^*$, in (s_i^*, s_{-i}) the number of users of e is x_e+1 , we have: (recall that $x_e^* = |\{i \mid e \in s_i^*\}|$)

$$\sum_{i=1}^{k} C_i\left(s_i^*, s_{-i}\right) = \sum_{i=1}^{k}\sum_{e \in S_i^*} C_e\left(s_i^*, s_{-i}\right) \leq \sum_{i=1}^{k}\sum_{e \in S_i*}\left(a_e\left(x_e+1\right)+b_e\right) =$$

$$= \sum_{e \in E}\left(a_e\left(x_e+1\right)+b_e\right)\cdot x_e^* \leq \sum_{e \in E}\frac{5}{3}\left(a_e x_e^* + b_e\right)x_e^* + \sum_{e \in E}\frac{1}{3}\left(a_e x_e + b_e\right)x_e$$

$$= \tfrac{5}{3}C(\mathbf{s}^*) + \tfrac{1}{3}C(\mathbf{s})$$

Corollary 5.4 *POA* $\leq \frac{5}{2}$

Corollary 5.5 *Smoothness is stronger than POA, since to derive a POA bound, it is only needed that*

$$\sum_i C_i\left(\mathbf{s}, \mathbf{s}^*\right) \leq \lambda \cdot \text{cost}(\mathbf{s}^*) + \mu \cdot \text{cost}(\mathbf{s})$$

in a special case where $\mathbf{s}$ *is a Nash equilibrium and* $\mathbf{s}^*$ *is optimal. In smoothnes, the inequality should be held for every pair of* $\mathbf{s}, \mathbf{s}^*$*, not only where* $\mathbf{s}$ *is a Nash equilibrium.*

5.2.6 The need for robustness

POA bound means that if the game is at a NE, then the outcome is near-optimal.

Sometimes we cannot reach an equilibrium since:

- Pure equlibrium may not exist
- It's hard to compute the equilibrium, even centrally
- It might be hard to learn in a distributed way.

Definition Robust POA bounds are worst-case bounds that apply even to non-equilibrium outcomes.

5.2.7 Beyond pure Nash Equilibria

Definition Mixed Nash Equilibria $\sigma = \sigma_1 \times .. \times \sigma_k$ (a multiplication of probabilites)

$\forall s, s'_i : E_{s\sim\sigma}[C_i(s)|s_i] = E_{s_{-i}\sim\sigma_{-i}}[C_i(s', s'_{-i})|s_i]$

A mixed NE exists when every player chooses his actions out of a distribution and given other players' action, he wouldn't want to deviate from his action.

Definition Correlated Nash Equilibria $\sigma \neq \sigma_1 \times .. \times \sigma_k$

$\forall s, s'_i : E_{s\sim\sigma}[C_i(s)|s_i] = E_{s\sim\sigma}[C_i(s', s'_{-i})|s_i]$

- Give a public strategy vector where each player can see only his action
- Every player is taking into account every possible action other player can make

A correlated NE exists if no player would like to deviate from his strategy (given other don't deviate).

Definition Coarse Correlated Nash Equilibria

$\forall s, s'_i : E_{s\sim\sigma}[C_i(s)] = E_{s\sim\sigma}[C_i(s', s'_{-i})]$

Like correlated NE but every player has a chance to choose his action in the public action vector or commit to playing the action being drawn.

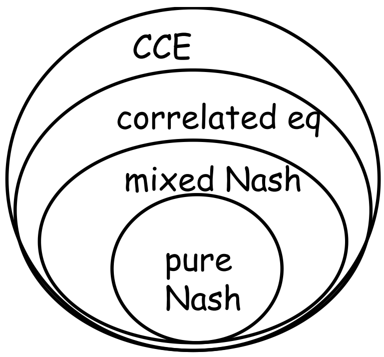

Figure 5.3: The containment of equilibriums

5.2.8 Price of Anarchy Bound in Repeated Play

Definition A sequence $s_1, s_2, \dots, s_T$ of outcomes is *no-regret* if for each player i, each fixed action q_i average cost player i incurs over sequence no worse than playing action q_i every time.

Example - The Expert Problem - There are k radio stations, and in each round $1 \leq t \leq T$:

- Each radio station predicts the weather.
- Each player should decide whether to take an umbrella or not.
- The loss is $\ell \in [0, 1]$

Claim 5.6 *There is an algorithm that solves the Expert Problem with regret $\sqrt{T}$ per round (or $\frac{\sqrt{T}}{T} = o(1)$ in all rounds). (The regret is relative to playing the best fixed action in all rounds.)*

Claim 5.7 $\{$*Coarse correlated equilibria (CCE)*$\}$ = $\{$*All probability distributions which are the limit of the empirical distribution of some no-regret sequence*$\}$

Theorem 5.8 [Roughgarden STOC 09] *In a (λ, μ)-smooth game, average cost of every no-regret sequence at most$\left[\frac{\lambda}{1-\mu}\right]$ $\times$cost of optimal outcome. (That is, in this case* POA $= \left[\frac{\lambda}{1-\mu}\right]$*)*

Proof Assume $\mathbf{s}^1, \mathbf{s}^2, \cdots \mathbf{s}^T$ is the no-regret sequence and $\mathbf{s}^*$ is the optimal outcome.

$$\sum_t \text{cost}\left(\mathbf{s}^t\right) = \sum_t \sum_i C_i\left(\mathbf{s}^t\right) = \sum_t \sum_i \left[C_i\left(s_i^*, s_{-i}^t\right) + \Delta_{i,t}\right]$$

Where $\Delta_{i,t}$ is defined in the following way:

$$\Delta_{i,t} := C_i\left(\mathbf{s}^t\right) - C_i\left(s_i^*, s_{-i}^t\right)$$

Therefore,

$$\sum_t \text{cost}\left(\mathbf{s}^t\right) \leq \sum_t \left[\lambda \cdot \text{cost}\left(\mathbf{s}^*\right) + \mu \cdot \text{cost}\left(\mathbf{s}^t\right)\right] + \sum_i \sum_t \Delta_{i,t}$$

Due to the no regret condition, $\sum_t \Delta_{i,t} \leq 0$ for each i, so after division by $1 - \mu$ and by T we will get:

$$\frac{1}{T}\sum_{t=1}^{T} \text{cost}\left(\mathbf{s}^t\right) \leq \frac{1}{T}\sum_{t=1}^{T} \frac{\lambda}{1-\mu} \text{cost}\left(\mathbf{s}^*\right)$$

Corollary 5.9 *The same bound also applies to Mixed NE and Correlated NE games.*

Proof Immidiate from Claim 5.7 and Theorem 5.8.

Theorem 5.10 [Roughgarden STOC 09] *For every set C, unweighted congestion games with cost functions restricted to C are tight:*

$$\max_{\substack{\text{congestion games} \\ \text{with cost fun. } C}} [\,\text{pure POA}\,] = \min_{\substack{(\lambda,\mu)\text{: all such games} \\ \text{are } (\lambda,\mu)\text{-smooth}}} \left[\frac{\lambda}{1-\mu}\right]$$