Coalitional Bargaining in Networks *

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Abstract

We analyze an infinite horizon, non-cooperative bargaining model for TU games with general coalitional structure. In each period of the game an opportunity for a feasible coalition to form arises according to a stochastic process, and a randomly selected agent in the coalition makes a take-it-or-leave-it offer. Agents that reach an agreement exit the game and are replaced by clones. We characterize the set of stationary equilibria by a convex program. We examine the implications of this characterization when the feasible coalitions are determined by an underlying network. We show how an agent's payoff is related to the centrality of his position in the network.

Keywords: Non-cooperative Bargaining, Coalition Formation, Network Games.

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1 Introduction

In the study of economic models of trade, it has long been recognized that institutional or physical barriers limit direct trade between some pairs of agents. Network models have been used to study such scenarios since the early 1940s, but only recently did they become objects of active research.[1] Agents in these models are represented by the nodes of a network, and the presence or absence of an link indicates the possibility or impossibility of trade between the relevant pair of agents. A goal of this line of work is to examine the impact of such barriers to trade and understand how an agent's position in this network influences his share of resources.

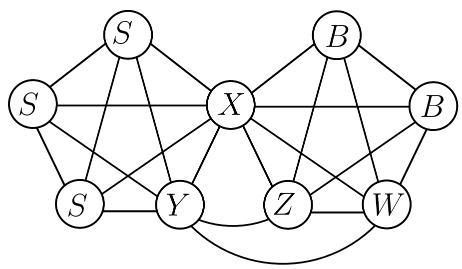

Figure 1: A trading network with intermediaries

As an illustration of the complexities that present themselves, consider the network illustrated in Figure 1. The S nodes correspond to sellers and the and the B nodes represent buyers. No direct trade is possible. X, Y, Z and W are intermediaries in this economy. X connects the market and bargains for a share of the trading surplus, but his bargaining power is affected by Y, Z and W. However, either Y and Z or Y and W need to cooperate in order to connect the buyers and the sellers. At the same time, Z and W compete for this cooperation. Given this network, which trading paths will be selected and how do the agents share the trading surplus?

Solution concepts from cooperative games like the Core (Shapley [1953]), the

[1]The literature goes back to the works of Kantorovich and Koopmans. See Schrijver [2005].

Myerson-Shapley value (Myerson [1977]) etc.. provide a variety of ready made answers to this question. However, they are not entirely satisfactory if one believes the interactions between the agents to be non-cooperative.[2]

In this paper we follow Rubinstein and Wolinsky [1985] and Manea [2011] and assume that payoffs are determined by a non-cooperative bargaining process. We develop a general approach using convex programing for these types of problems. The approach overcomes the complexity of previous models, such as Manea [2011]. Furthermore, it allows for several extensions that capture intermediaries and multilateral trade in bargaining.

For example, given the network formulation described above, a seller and a buyer need one or more intermediaries that form a connecting path to trade. Each trading path can be thought of as a coalition among a subset of agents. To capture the complex structure of these coalitions we consider a TU game with an arbitrary set of *feasible* coalitions. Our goal is to understand how an agent's position in the collection of feasible coalitions effects their bargaining power.

The game is defined as follows. Given a TU game with a set of feasible coalitions, bargaining is modeled as an infinite horizon discrete time game, where each agent has an individual discount factor. In each period a feasible coalition is picked according to a stationary distribution, then an agent in the coalition is chosen at random to be the proposer.[3] The proposer can decline to propose or sequentially suggest a division of the coalition surplus to its members. If the proposer declines to propose or at least

[2]Non-cooperative bargaining processes have been studied for cooperative games, for example Gul [1989], Hart and Mas-Colell [1996]. However, these results do not extend to network models. A detail discussion is provided in the review of related literature.

[3]The stochastic process can be generalized so that the probability that an agent i is the proposer for a coalition S is of the form $\alpha_i \cdot P(S)$. This model can capture a situation where there is a steady population of agents of type i, and $\alpha_i \cdot P(S)$ corresponds to a general matching function that depends on the population distribution. One can also interpret α_i as the probability that agent i searches and $P(S)$ as the probability that S is found. More discussion about this is provided in Section 2.

one member of the coalition disagrees with the suggested share, the game continues to the next stage where another coalition is selected. Otherwise, agents in the coalition exit the game with their proposed division, and are replaced by a clone.[4] The game repeats.

We show that in this model a stationary equilibrium exists, and its payoff is the *unique* solution of a strictly convex program. A constraint of this program corresponds to a feasible coalition, and if it is slack in the optimal solution, then the corresponding coalition never forms at equilibrium. In contrast with the core of the same TU game, which is characterized by a *linear* program, our solution always exists and is unique. Hence, it provides a robust prediction on the prices and pattern of trade that will emerge in the bargaining process.

Moreover, while in the static TU game the primal and the dual variables of the linear program correspond to Walrasian prices and the associated trade pattern, in our model the primal-dual relationship of the convex program can be interpreted as a qualitative connection between an agent's bargaining power to his credible outside options. The intuition is the following. As we will see, according to the convex program an agent's payoff is proportional to the sum of the duals corresponding to the coalitions that the agent is a member of. If a constraint is slack, its dual is 0, and thus, it does not influence the agents' payoff. On the other hand, due to our characterization, a coalition corresponding to a slack constraint never forms at equilibrium. Hence, the primal-dual connection explicitly captures the property that a coalition which never forms in equilibrium can not be used as a credible outside option

[4] The assumption that agents are faced with a stationary distribution of bargaining opportunities, and if they exit, they are replaced by clones captures a steady state scenario in which the flow of departures is matched by an equal arrival flow of agents in each type. This is a standard assumption in search theoretic models. Papers that explicitly model bargaining protocols in similar environments include Rubinstein and Wolinsky [1985], Gale [1987] and Manea [2011]. However, these papers only consider pairwise bargaining.

in bargaining. This generalizes the insight of Binmore et al. [1989] that "outside options should not affect an agent's bargaining power if they are not credible".

We apply the coalitional model to several network bargaining games, including buyer-seller networks, trading networks with intermediaries and cooperation in overlapping communities. In each of these games, we use the payoff an agent receives as discount rates go to 1 to understand how position in the network can influence the distribution of resources. These payoffs will be used as an index of bargaining power. We contrast our solution with the core of the corresponding TU game and use the convex program to examine how an agent's bargaining power depends on the centrality of his position in the network.

1.1 Related Literature

At a high level our framework resembles the literature on non-cooperative bargaining in TU games. However, prior work along this line assumes that the market is static, whereas in our model we assume agents arrive to and depart from the market. Moreover, in contrast with our paper, these works assume that all coalitions are feasible.

In particular, previous work on non-cooperative bargaining in TU games assumes that agents are not replaced by a clone after trade, and the objective is to identify a bargaining process that will provide a non-cooperative foundation for a standard solution concept like the Shapley value. Examples of such contributions are Gul [1989] and Hart and Mas-Colell [1996]. Our model captures a flow-market and assumes the economy is in a steady state. Here, an agent is always replaced by a clone in the future, thus, the structure of the game does not change over time. On the contrary, for example, in Hart and Mas-Colell [1996] it is assumed that with a small probability a proposer can be rejected from the game and receives 0 payoff. In such a game all coalitions play a role in equilibrium because there is always a chance that a given set of agents have disappeared from the game, which explains why its outcome can coincide with the Shapley value.

1.2 Outline of The Paper

Section 2 introduces the coalitional bargaining model and characterizes the stationary equilibrium payoff of the coalitional bargaining model as a convex program.

Section 3 discusses the applications of coalitional bargaining to network games. In particular, Section 3.1 examines a buyer-seller network where a feasible coalition is a pair of agents that are connected by a link. Here we compare our convex program with the solution of Shapley and Shubik [1972], which is characterized by a linear program.

Section 3.2 gives a new and simpler proof for the results of Manea [2011], which is a special case of our model.

Section 3.3 discusses a new application of trading networks with intermediaries, where we model each feasible coalition as a path connecting a buyer and a seller. We show that an agent's bargaining power depends on the centrality of the agent's position in the network. Furthermore, the measurement of centrality coincides with its level of congestion in an associated congestion game.

Section 3.4 models cooperation in overlapping communities. Here we define the feasible coalitions to be the set of cliques of a given network. Hence, only set of agents that belong to the same community or close to each other can cooperate. In this game the structure of communities and hierarchy emerges endogenously. We show that an agent at the intersection of communities has high bargaining power, which captures the social structural holes studied in Burt [1992].

Section 4 concludes. Most of the proofs are given in the appendix.

2 Coalitional Bargaining

2.1 The model

Suppose a set of n agents, $N = \{1, \ldots, n\}$. Let the family $\mathcal{F} \subset 2^N$ denote the set of feasible coalitions. The surplus that a coalition $S \in \mathcal{F}$ generates is non-negative and denoted by $V(S)$.

Consider the following infinite horizon bargaining game. In each period $t = 0, 1, \ldots$ a coalition $S \in \mathcal{F}$ and one of the agents $i \in S$ is selected randomly with a probability of the following form

$$Pr(S,i) = \alpha_i \cdot P(S), \text{where } \alpha_i, P(S) > 0.$$

There are several interpretations of this form of probability distribution, which will be described below.

Agent i can either *sequentially* make an offer to the other agents in S specifying a division of $V(S)$, or decline to make such an offer.[5] If agent i declines or at least one agent in S rejects the offer made by i, all agents remain in the game for the next period. Otherwise, they exit the game with their agreed upon shares and are replaced by a clone. In period $t+1$, the game is repeated. For each i, $\delta_i \in (0,1)$ is the discount

[5]Sequential offer in coalitional bargaining is a standard way to prevent the equilibrium that all agents refuse the offer in a simulations offer game.

factor of agent i. The game is denoted by

$$\Gamma(N, \mathcal{F}, V(.), P(.), \vec{\alpha}, \vec{\delta}).$$

The process of selecting a coalition and a proposer with probability $\alpha_i \cdot P(S)$ is quite general. One can think of this stochastic matching process in a search-theoretic framework. The parameter α_i represents agent i's searching ability. Independently, $P(S)$ is the probability that a coalition S is found. Once i finds S, i is the proposer.

Another scenario can be that S is selected at random with probability $Pr(S)$ and among members of S a proposer is picked at random. In this case one can take $\alpha_i = 1, P(S) = \frac{Pr(S)}{|S|}$. This process can be extended to the case where an agent is chosen to be the proposer with probability proportional to a given weight $\gamma_i \geq 0$, that is, the probability that i is selected given the coalition S is $\frac{\gamma_i}{\sum_{j \in S} \gamma_j}$. Here, we choose $\alpha_i = \gamma_i, P(S) = \frac{Pr(S)}{\sum_{j \in S} \gamma_j}$.

Another example is the model of stationary markets in the spirit of Gale [1987] and Manea [2011]. Here, we assume that for each i there is a continuum of agents of type i with a positive measure μ_i. For each coalition S a measure β_S corresponds to the volume of possible trade for agents whose types give the set S. An interpretation is that an agent i represents a seller or a buyer of a product with a specific parameter i. Different combinations of products give a different value, which is captured by $V(S)$ for every feasible set S. We assume that all agents of the same type are treated symmetrically, furthermore, for simplicity we also assume that given a coalition S, the probability of choosing an agent i to be the proposer is the same for all members of S. In this case, the probability that a coalition S and a proposer $i \in S$ to be chosen is

$$\frac{1}{\mu_i} \cdot \frac{\beta_S}{|S|}.$$

In this paper, we will focus on the solution concept of *stationary equilibrium*: a subgame perfect equilibrium, where each agents strategy at any time depends exclusively on his identity and the play of the game: which coalition is chosen and which agent is the proposer and what he proposes.

More precisely, given an agent i, assume that at time t, a coalition S is chosen, agent k is selected as the proposer. First, if $k = i$, then a stationary strategy σ_i maps S to an element of $\mathbb{R}^{|S|-1} \cup \{\varnothing\}$, where $\varnothing$ corresponds to agent i's declining to offer and an $\vec{x} \in \mathbb{R}^{|S|-1}$ is his offer to other members of the coalition. Second, if $k \neq i, i \in S$, then σ_i maps $(S, \vec{x})$ to a probability $0 \leq p \leq 1$ of accepting the offer $\vec{x} \in \mathbb{R}^{|S|-1}$ from k. If $i \notin S$, then σ_i is irrelevant.

2.2 Equilibrium Characterization

We now characterize the unique stationary equilibrium payoffs of $\Gamma(N, \mathcal{F}, V(.), P(.), \vec{\alpha}, \vec{\delta})$ as the optimal solution of a convex program (Theorem 1). This result allows us to characterize the set coalitions that form in an equilibrium. Finally, we analyze the equilibrium payoff when agents are patient (Theorem 2).

THEOREM 1 *Given the game $\Gamma(N, \mathcal{F}, V(.), P(.), \vec{\alpha}, \vec{\delta})$, the payoff of any stationary equilibrium is the unique optimal solution of the following convex program*

$$
\begin{aligned}
\textit{minimize:} \quad & \sum_{i \in N} \frac{\delta_i(1-\delta_i)}{\alpha_i} u_i^2 + \sum_{S \in \mathcal{F}} P(S) z_S^2 \\
\textit{subject to:} \quad & \sum_{i \in S} \delta_i u_i + z_S \geq V(S) \quad \forall S \in \mathcal{F}, \\
& u_i \geq 0, z_S \geq 0 \;\; \forall i \in N, S \in \mathcal{F}.
\end{aligned} \tag{A}
$$

Furthermore, let u^A be the optimal solution of (A), then a stationary equilibrium is characterized as follows. When a coalition S is selected and $i \in S$ is the proposer, *then with probability 1:*

- *if* $\sum_{j \in S} \delta_j u_j^A < V(S)$*, i offers $\delta_j u_j^A$ to every $j \in S$ and j accepts,*
- *if* $\sum_{j \in S} \delta_j u_j^A > V(S)$*, either i refuses to make an offer or least one agent $j \in S$ rejects.*

Proof. See Appendix A.1.

Remark. The main idea of the proof is to interpret the condition for $\vec{u}$ to be a stationary equilibrium payoff as a first-order condition of the convex program above.

Theorem 1, however, does not imply uniqueness of equilibrium strategies. For instance, if $\sum_{j \in S} \delta_j u_j^A > V(S)$ and agent i is the proposer, all possible sharing schemes that leave i at least $\delta_i u_i^A$ are in equilibrium because for such sharing schemes at least one agent $j \in S$ will be offered less than $\delta_j u_j^A$ and the coalition will be rejected. However, this is an artificial issue: if a coalition is rejected, then the specific offer that results in rejection does not matter.

In our repeated game, a feasible coalition can be thought of as an outside option of the agents in the bargaining process. However, an important distinction is that a coalition that is always rejected in equilibrium cannot be used as a credible threat. Theorem 1 shows that a coalition whose corresponding constraint in (A) is slack is rejected and never forms.

Notice that in the convex program above if α_i increases, then the coefficient of u_i decreases. As a result, in the optimal solution the value u_i increases. This captures the intuition that when an agent has more chance to propose, he gets a higher payoff.

2.3 When Agents Are Patient

In the applications to network games, to isolate the effect of network structure on agents' bargaining power, we will assume that agents are symmetric in terms of the probability being chosen as a proposer and the discount factor. Namely, we assume $\alpha_i = 1$, $\delta_i = \delta$ for all i. This game is denoted by Γ^δ. As shown above the unique stationary equilibrium payoff is characterized by

$$\begin{aligned} &\min \sum_{i \in N} u_i^2 + \frac{1}{\delta(1-\delta)} \sum_{S \in \mathcal{F}} P(S) z_S^2 \\ &\text{subject to } \sum_{j \in S} \delta u_j + z_S \geq V(S) \quad \forall S \in \mathcal{F}, \\ &u_i \geq 0, z_S \geq 0 \;\; \forall i \in N, S \in \mathcal{F}. \end{aligned} \tag{A'}$$

We are interested in the outcome of Γ^δ as δ approaches 1. In this case the optimization program (A') approaches the following program

$$\begin{aligned} &\text{minimize } \sum_{i \in N} u_i^2 \\ &\text{subject to} \sum_{j \in S} u_j \geq V(S) \quad \forall S \in \mathcal{F}. \end{aligned} \tag{B}$$

Note that when agents are patient, $P(S)$ is irrelevant to the outcome of the game. The intuition is that agents can always wait for the best trading opportunity as long as the probability of that happening is positive.

The following theorem establishes convergence of Γ^δ as $\delta \to 1$.

THEOREM 2 *Let $u^{*\delta}$ be the outcome of Γ^δ (optimal point of (A')) and u^* be the solution of (B), then $\lim_{\delta \to 1} u^{*\delta} = u^*$.*

Furthermore, there exists $0 < \delta_0 < 1$ such that if a set $S \in \mathcal{F}$ satisfying $\sum_{i \in S} u_i^* > V(S)$, then the coalition S never forms at equilibria of the game Γ^δ for all $\delta \geq \delta_0$.

Proof. See Appendix A.2.

Finally, for the applications in the coming sections, we will need the following lemmas. They follow from the KKT conditions. The proof is standard, for example, see Rockafellar [1970].

LEMMA 2.1 *If $u^{*\delta}$ is the optimal solution of (A'), then there exist $\lambda_S^\delta \geq 0$ for all $S \in \mathcal{F}$ such that $\lambda_S^\delta = 0$ for every S, whose corresponding constraints in (A') does not bind and*

$$u_i^{*\delta} = \sum_{S \in \mathcal{F}: i \in S} \lambda_S^\delta \ \ \forall i \in N.$$

LEMMA 2.2 *u^* is the optimal solution of (B) if and only if u^* is feasible and there exist $\lambda_S \geq 0$ for all $S \in \mathcal{F}$ such that*

$$u_i^* = \sum_{S \in \mathcal{F}: i \in S} \lambda_S \ \ \forall i \in N,$$

$$\lambda_S (\sum_{i \in S} u_i^* - V(S)) = 0.$$

Remark. Lemma 2.2 can be interpreted as a duality between bargaining power and credible outside options. In particular, u_i^* serves as the index of agent i bargaining power, and a coalition that never forms can be considered as an incredible outside option. Notice that the incredible outside options correspond to the slack constraints at the optimal solution. Now, according to the duality lemma, there exists $\lambda_S \geq 0$ for each S such that $u_i^* = \sum_{S \in \mathcal{F}: i \in S} \lambda_S \ \ \forall i \in N$. Furthermore, if the constraint S is slack, then $\lambda_S = 0$. This means that an incredible outside option does not effect an agent's bargaining power, which generalizes the insight of Binmore et al. [1989].

3 Applications of Coalitional Bargaining Model

In this section we consider four applications of the coalitional bargaining model and its characterization developed in the previous section. We assume agents have the same chance of proposing, that is, $\alpha_i = 1$, furthermore, they have a common discount factor δ and $\delta \to 1$. Notice that when $\delta \to 1$ the outcome of the game does not depend on the stochastic matching function. Hence, an agent's payoff only depends on the structure of the feasible coalitions. The payoffs in this specification will be used as an index of bargaining power. In the applications below, they are determined by the topology of the underling network.

First, in Section 3.1 we apply the bargaining framework to the assignment model, which was introduced by Shapley and Shubik [1972], so that we can contrast the predictions of this model to that from the assignment game. Second, in Section 3.2 we apply it to the model of Manea [2011] to give new and simpler proof for some of his results. The third application, analyzed in Section 3.3, is to a model of a buyer-seller network with intermediaries. Here we show that an agent's bargaining power depends on the centrality of his position in the network. Our convex program rigorously define the centrality of a node in the network. Furthermore, we show that this measurement coincides with level of congestion in an associated traffic network. Finally, in Section 3.4 we examine a game of cooperation in overlapping communities, which is previously studied by Kets et al. [2011]. Our result shows that a hierarchical structure emerges endogenously. Furthermore, an agent at the intersection of communities has high bargaining power, which captures the theory of social structural holes studied in Burt [1992].

3.1 Shapley-Shubik Assignment Model

Given a seller-buyer network (a bipartite graph), let $V_{ij} \geq 0$ be a nonnegative surplus for each pair of buyer-seller (i, j). For a subset of sellers and buyers, the maximum surplus can be generated by these agents is the value of a maximum matching among the agents. Shapley and Shubik show that the core of this game is non empty and corresponds to the solutions of the linear programming dual to the optimal assignment problem.

$$\text{minimize: } \sum_i u_i$$

$$\text{subject to: } u_i + u_j \geq V_{ij}.$$

Note that the solution of this program is not necessarily unique. It can be understood as Walrasian prices that balance supply and demand in this two-sided market.

In our bargaining game a feasible coalition correspond to a buyer-seller pair (i, j) whose surplus is the given value V_{ij}. Note that as $\delta \to 1$ if trade between a pair of agents is not "efficient enough", one of the two agents might want to wait for better trading opportunities. One might suspect that the outcome of the bargaining game will be efficient. However, the convex program is different from the optimal assignment program of Shapley and Shubik, given above. Our program has the same feasible region but different objective function which has a unique solution. Specifically:

$$\text{minimize: } \sum_i u_i^2$$

$$\text{subject to: } u_i + u_j \geq V_{ij}.$$

Note that the pairs of agents that trade in both models correspond to the binding constraints in an optimal solution of the relevant program. Thus, as illustrated in the examples below, the outcomes are qualitatively different. The two programs can be interpreted as giving different predictions of the prices and pattern of trade that will emerge given the same supply-demand data. In the Shapley-Shubik framework the game is one-shot and the data is static. In our model the data is a snapshot of a flow economy.

To illustrate, consider a simple example of one seller S and two buyers B_1, B_2, the trade surplus between S, B_1 and S, B_2 are 1 and a, respectively. We assume that $a > 1$. In the Shapley-Shubik's solution only S and B_2 trade. In the bargaining game, there are two cases. (i) If $a > 2$, then only S and B_2 trade and they both share half of the surplus: $\frac{a}{2}$. (ii) If $1 < a \leq 2$, then both pairs can trade, depending on which one is chosen first, and the seller's payoff is $x = \frac{a+1}{3}$, B_1 obtains $\frac{2-a}{3}$ and B_2 gets $\frac{2a-1}{3}$ in the bargaining game.

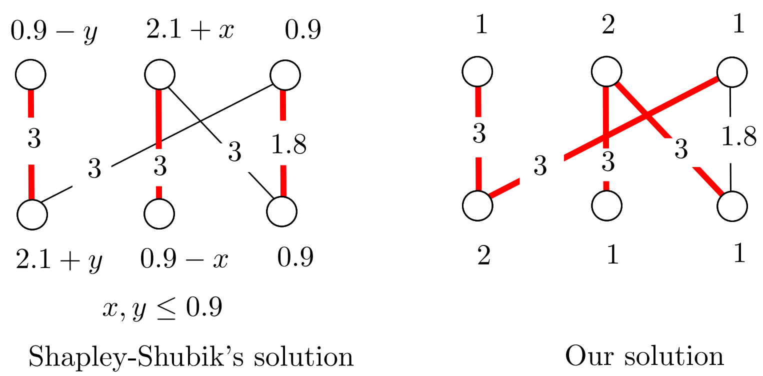

Figure 2: Trade pattern of two models is different.

Another example is illustrated in Figure 2. The surplus between two nodes are listed on the link connecting them. A missing link represents a surplus of 0. In the figure on the left, the three bold links form the maximum matching and the number next to a node is its payoff determined by Shapley-Shubik program, which is unique in this case. The figure on the right shows the outcome of our model. The payoffs are unique and unlike the Shapley-Shubik solution, the pair with the surplus of 1.8

do not trade.

The two trading patterns coincide for rather special cases, which are characterized by the following result.

PROPOSITION 3.1 *Given an assignment model if the efficient assignment is unique and denoted by M, then the trading pattern of the bargaining game (the set of seller-buyer pairs that trade) is exactly M if and only its unique payoff u^* satisfies*

$$u_i^* = u_j^* = \frac{V_{ij}}{2} \text{ if } (i,j) \in M,$$

$$u_i^* = 0 \text{ if } i \text{ is not covered by } M, \text{ and}$$

$$u_i^* + u_j^* > V_{ij} \text{ if } (i,j) \notin M.$$

Proof. See Appendix A.3.

3.2 Manea's Model

A special case of our model has been studied by Manea [2011]. Here, given a undirected network G with the set of links denoted by E, the feasible coalitions are the pairs $(i,j) \in E$. The value of each coalition is one unit. To analyze this game, Manea introduces several new concepts, such as mutually estranged sets, partners and shortage ratios. We show that in our framework these concepts and his results emerge naturally from the primal-dual relationship of the following convex program.

$$\begin{aligned} &\text{minimize: } \sum_i u_i^2 \\ &\text{subject to: } u_i + u_j \geq 1 \;\; \forall (i,j) \in E. \end{aligned} \tag{1}$$

By Lemma 2.2, a feasible u^* is the optimal solution of (1) if and only if there exists $\lambda_{ij} \geq 0$ for all $(i, j) \in E$ such that

$$\begin{aligned} u_i^* &= \sum_j \lambda_{ij}, \\ \lambda_{ij}(u_i^* + u_j^* - 1) &= 0. \end{aligned} \tag{2}$$

Let G^* be the sub network of G that consists of the links (i, j) for which $u_i^* + u_j^* = 1$. The results of Manea [2011] essentially can be stated as follows.

PROPOSITION 3.2 *(Theorem 3 of Manea [2011]) Let M be an independent set in G^*, let L be the set of partners of M, i.e, the set of nodes that is connected to M by a link in G^*, then*

$$\min_{i \in M} u_i^* \leq \frac{|L|}{|L| + |M|} \text{ and } \max_{j \in L} u_j^* \geq \frac{|M|}{|L| + |M|}.$$

Proof. Because M is an independent set in G^*, $u_{i_1}^* + u_{i_2}^* < 1$ for $i_1, i_2 \in M$. Because of (2) $\lambda_{i_1 i_2} = 0$, and thus,

$$\sum_{i \in M} u_i^* = \sum_{ij: i \in M, j \in L} \lambda_{ij} \leq \sum_{j \in L} u_j^*.$$

The last inequality holds because nodes in L might have partners outside M. Now, because for every $i \in M$, there exists $j \in L$ such that $u_i^* + u_j^* = 1$, and vice versa, thus,

$$\min_{i \in M} u_i^* = 1 - \max_{j \in L} u_j^*.$$

Let $a = \min_{i\in M} u_i^*$, we have

$$|M|\cdot a \le \sum_{i\in M} u_i^* \le \sum_{j\in L} u_j^* \le |L|\cdot(1-a).$$

Therefore, $a \le \frac{|L|}{|L|+|M|}$ and $1-a \ge \frac{|M|}{|L|+|M|}$. ∎

PROPOSITION 3.3 *(Theorem 4 and 5 of Manea [2011]) Let H be a connected component of G^*, then one of the followings is true.*

- *H is a bipartite graph whose two classes are L, M and*

$$u_i^* = \frac{|L|}{|L|+|M|} \text{ for } i\in M \text{ and } u_j^* = \frac{|M|}{|L|+|M|} \text{ for } j \in L.$$

- *H is non bipartite and $u_i^* = 1/2$ for all $i \in H$, furthermore, H can be covered by a disjoint union of odd length cycles and matchings.*

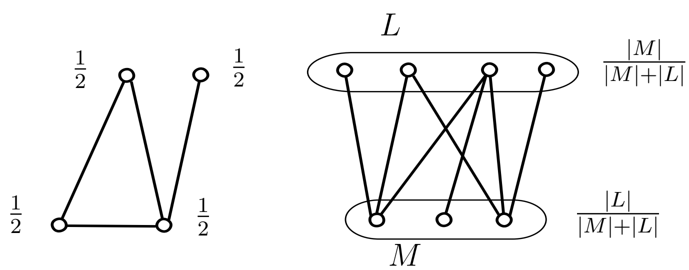

Figure 3: A payoff characterization

Proof. First, consider the case when H is a bipartite graph with two sides L and M. See Figure 3. Because H is a connected component of G^*, there exists $a > 0$ such that $u_i^* = a$ for all $i \in M$, and $u_j^* = 1-a$ for all $j \in L$.

Because there are no links in G^* connecting H and $G - H$, we have $u_k^* + u_l^* > 1$ for $k \in H, l \in G - H$. Therefore, there exists $\epsilon > 0$ such that by increasing (decreasing) all $u_i^*, i \in M$ by ϵ and decreasing (increasing) all $u_j^*, i \in L$ by ϵ, the new solution does not violate any constraint of (1).

Now, $u_i^* = a$ for all $i \in M$ and $u_j^* = 1 - a, j \in L$ is in the optimal solution of (1). Consider the value

$$\sum_{i \in M} u_i^{*2} + \sum_{j \in L} u_j^{*2} = |M|a^2 + |L|(1-a)^2.$$

Because we can both increase a decrease a by $\epsilon > 0$, the value above is optimal if the derivative of $|M|a^2 + |L|(1-a)^2$ is zero. This gives $a = \frac{|L|}{|L|+|M|}$.

Second, if H is a non-bipartite graph, then consider an odd length cycle of H. Because $u_i^* + u_j^* = 1$ holds for every link i, j along the odd length cycle, we have $u_i^* = 1/2$ for all $i \in H$.

Now because of (2), there exists $\lambda_{ij} \geq 0$ for all links ij in H such that

$$\frac{1}{2} = \sum_{j \in H} \lambda_{ij} \text{ for all } i \in H.$$

It is well known that this condition is equivalent to the fact that H can be covered by odd cycles and matching, see for example Lovász and Plummer [1986]. ∎

3.3 Trading Networks with Intermediaries and Network Centrality

We now examine a model of trading network with intermediaries, which is based on the model of coalition bargaining. Here the feasible coalitions are the paths connecting buyers and sellers.

This application is different from the ones discussed previously because here trade involves intermediaries. Notice that when there are no intermediaries, equilibrium payoffs can be interpreted in terms of supply-demand theory.[6] In a trading network with intermediaries, we do not have a similar interpretation because intermediaries do not produce nor consume. Here an agent's bargaining power depends on how central his position is in the network. Our main finding is that in this setting, the notion of network centrality can be defined and computed rigorously. Furthermore, this definition coincides with the well known concept of congestion level in an associated traffic network. In the following we introduce the notation for the bargaining game and use it to define the associated traffic network.

In particular, we are given a network (either directed or undirected) denoted by G and a set of sellers-buyer pairs $\{(s_k, b_k) : k \in K\}$, each is associated with a nonnegative trading surplus V_k.[7] The bargaining game is defined as in Section 2, where a feasible coalition is a set of agents who form a path connecting a seller s_k and a buyer b_k, and its value is V_k. Here we also assume that agents have the same discount rate δ. Furthermore, given a trading path S all agents in S has the same probability of being the proposer, which is given by $P(S)$. (This is a special case of the coalitional bargaining model where $\alpha_i = 1$ for all i.) The game is denoted by

$$\mathcal{B}(G, \{(s_k, b_k, V_k) : k \in K\}, P, \delta).$$

Our main finding is that an agent's bargaining power coincides with its congestion level in an associated congestion game that models a traffic network where vehicles

[6]The solution of Shapley and Shubik [1972] and Manea [2011] can be interpreted as prices that balances supply and demand. In particular, the argument to establish the limit payoff of Manea [2011] and the ascending pricing process of Kranton and Minehart [2001] for Shapley-Shubik's solution are based on variants of the Hall marriage theorem. A crucial steps in these processes is to find a subset of agents such that the ratio between its size and the number of their trading partners is small.

[7]This general framework can model trade with heterogeneous goods, where a seller can only trade with some specific buyers. Homogeneous trade is a special case where the set $\{(s_k, b_k) : k \in K\}$ contains all possible seller-buyer pairs.

are the agents that choose a route with the smallest congestion.

Bargaining power in an intermediary network can be intuitively understood as follows. An intermediary charges a price for trade going through him. Such a price is determined by a bargaining process and thus, a "high" price is only sustainable if the agent has "many" credible outside options. Outside options of an agent are the trading paths that go through him. If an intermediary charges too high a price, sellers and buyers will choose an alternative path. This will reduce the intermediary's credible outside options and reduces what he can credibly demand.

Striking a balance between the credible outside options and the intermediary's price is reminiscent of congestion games. In particular, given a trading network $(G, \{(s_k, b_k, V_k) : k \in K\})$ as described above, its associated congestion game is defined as follows. In the underlying network G, treat each (s_k, b_k) as an origin-destination pair. The traffic on this network is modeled by the notion of *flow*, that is, a non-negative value on each path connecting an origin-destination pair. The flow of a path represents the amount of traffic traveling on it.

Given a path S let f_S be its flow. Denote by $\vec{f}$ the vector of flows on all possible paths. For a given $\vec{f}$, let $x_i(\vec{f})$ be the total traffic that going through node i, that is,

$$x_i(\vec{f}) = \sum_{P: i \in P} f_P.$$

The *congestion* at node i is equal to its total traffic $x_i(\vec{f})$. Given a path S and a traffic flow $\vec{f}$, the congestion of the path S is the total congestion of all nodes in S, and is denoted by

$$l(S, \vec{f}) = \sum_{i: i \in S} x_i(\vec{f}).$$

Given an origin-destination pair (s_k, b_k), we assume that any vehicle leaving s_k

for b_k is selfish and "nonatomic". The term nonatomic reflects the assumption that there is a continuum of vehicles and no single vehicle can influence the amount of traffic on any route. A vehicle is selfish because it chooses a path with the smallest congestion. We further assume that for each origin-destination pair (s_k, b_k), V_k is the maximum level of congestion that a vehicle going from s_k to b_k can tolerate and there is always demand for traveling between s_k and b_k if there exists a connecting path of congestion less than V_k. In other words, the demand for traveling from s_k to b_k as a function of total congestion is inelastic with the cutoff of V_k.

Hence, traffic flow $\vec{f}$ is in equilibrium if for any two paths S, S' connecting s_k, b_k

$$f_S > 0 \iff V_k = l(S, \vec{f}) \leq l(S', \vec{f}).$$

The congestion game described above is denoted by

$$\mathcal{T}(G, \{(s_k, b_k, V_k) : k \in K\}).$$

Figure 4 shows a particular instance of the congestion game associated with the trading network discussed in the introduction. The origins are the three nodes labeled "10" and the destinations are labeled "15". (These are the sellers and buyers of the trading network.) Traffic can tolerate up to a congestion of 43, which is the trading surplus between a buyer and a seller. The numbers at the nodes indicate their congestion level. The number on each bold link represents the traffic going from one node to the other. A directed path consisting of bold links corresponds to a traffic flow at an equilibrium.

It is well known that the congestion level at a node of the traffic network at an equilibrium of this congestion game is unique and corresponds to a solution of a

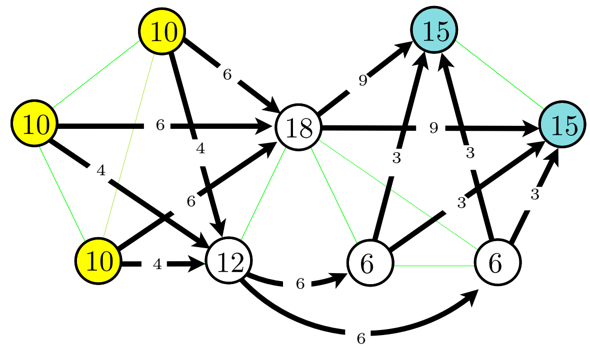

Figure 4: An instance of congestion game.

convex program. It turns out that this convex program is the same as the one in our bargaining model when agents are patient. This gives us the following result.

PROPOSITION 3.4 *As $\delta \to 1$, the stationary equilibrium payoff of the bargaining game $\mathcal{B}(\delta, G, \{(s_k, b_k, V_k) : k \in K\})$ converges to the congestion level on the nodes of G in the congestion game $\mathcal{T}(G, \{(s_k, b_k, V_k) : k \in K\})$.*

Proof. See Appendix A.4.

Remark. A special case of our game is where the surplus for every pair of nodes is their distance in the underlying network. The payoffs of the bargaining game reflects a measurement of centrality in network, which is similar to the well known concept of "betweenness centrality" introduced in Freeman [1979]. The betweenness centrality for a given node is defined as the number of shortest paths from all vertices to all others that pass through that node. Our solution is in the spirit of this measurement and our approach might provide a non-cooperative game theoretic foundation for network centrality, which have been defined and used in an ad-hoc manner.

3.4 Cooperation in Overlapping Communities

In this model, motivated by Kets et al. [2011], simultaneous trade among more than two agents is possible and modeled by a supermodular surplus function.[8] A feasible coalition is a clique in an underlying network. The motivation for such a scenario is that communication and geographical properties are barriers of trade and only agents who are close to one another geographically or belong a common community can trade.

The main finding is that hierarchical structures and overlapping communities emerge endogenously as an outcome of the bargaining game. Here, each community is defined as a *maximal* clique of the network and an agent can belong to multiple communities. The hierarchical structure in each community follows from the nesting property of the coalitions that form at equilibrium.

All Coalitions Are Feasible First, consider a special case of our network game where the underlying network is complete, i.e., every pair of nodes are connected. In this case all coalitions are feasible. Thus, our game is a bargaining process for a traditional TU game (N, V), where N is the set of agents and V is the characteristic function. We denote the bargaining game in this case by (N, V, δ) and let $u_i^{*\delta}$ be the unique payoff of agent i in this game. We have the following result.

PROPOSITION 3.5 *If the network is complete, then as $\delta \to 1$, $u_i^{*\delta}$ converges to the point in the core of the game (N, V) that has the minimum Euclidean norm.*

Furthermore, if the game is strictly supermodular, then there exist a set of coalitions $S_1 \supsetneq S_2 \supsetneq \cdots \supsetneq S_K$ and $0 < \delta_0 < 1$, such that for all $\delta > \delta_0$ there exist

[8] $f : 2^N \to \mathbb{R}^+$ is supermodular if $f(A) + f(B) \leq f(A \cup B) + f(A \cap B)$.

nonnegative $\lambda_1^\delta, \ldots, \lambda_K^\delta$, *where*

$$u_i^{*\delta} = \sum_{k:i \in S_k} \lambda_k^\delta,$$

and only coalitions in $\{S_k : k = 1, \ldots, K\}$ *form at equilibrium.*

Proof. See Appendix A.5.

The proposition above can be interpreted as follow. In strictly supermodular games, when agents are patient enough there is hierarchy structure defined by the set of coalitions that form at equilibrium. Agents that belong to a smaller coalition have higher rank and have higher bargaining power.

Coalitions Are Cliques of A Network We now consider the game in a general network denoted by G. Feasible coalitions are the cliques of G. Given a discount rate δ, we denote the game by $\mathcal{G}^\delta$ and let $u^{*\delta}$ be the unique stationary equilibrium payoff. Given a network G, we call a subset of nodes a *community* of G if they form a *maximal* clique.

We have the following result, whose proof is analogue to that of Proposition 3.5.

PROPOSITION 3.6 *Given a network* G, *let* $C_1, \ldots, C_M$ *be the set of communities of* G. *If* V *is strictly supermodular, then there exists* $0 < \delta_0 < 1$ *and a set of nested coalitions in each community* C_m: $S_1^m \supsetneq S_2^m \supsetneq \cdots \supsetneq S_{K_m}^m$, *such that for all* $\delta > \delta_0$ *there exist nonnegative* $\lambda_{k,m}^\delta$ *associated with the coalition* S_k^m *where*

$$u_i^{*\delta} = \sum_{m,k:i \in S_k^m} \lambda_{k,m}^\delta.$$

Furthermore, all other coalitions do not form in the bargaining game.

The interpretation of this result is that agents bargaining power depends on the combination of the rank in the hierarchy of each community and the variety of communities that he belongs to. Intuitively, agents in more communities have access to more trade opportunities and have higher bargaining power. Using our approach, we can decompose an agent's bargaining power into different factors captured by the variable λ.

The following proposition shows that when the characteristic function on each maximal clique only depends on the size of the coalition, then agents belonging to more communities will have higher payoff.

COROLLARY 3.7 *Assume $V(S)$ is a function of $|S|$. Consider agent i and j such that i belongs to all communities that j is a member of, then $u_i^{*\delta} \geq u_j^{*\delta}$.*

Proof. See Appendix A.6.

4 Conclusion

We have introduced a model of non-cooperative bargaining in TU games with a coalitional structure and provide a method using convex programming to characterize the stationary equilibrium. We discussed several applications to network bargaining. The method is general and can be extended, see for example Nguyen [2011].

A Missing Proofs

A.1 Proof of Theorem 1

Assume a stationary equilibrium exists. Let u_i be the expected payoff of agent i in any subgame of such equilibrium. When a set S is selected and i is chosen as the proposer, for $j \in S$, j's continuation payoff is $\delta_j u_j$ in the case of disagreement. Therefore, j accepts any offer of at least $\delta_j u_j$.

Thus, to get his offer accepted agent i will offer $\delta_j u_j$ to j, and obtain a payoff of

$$V(S) - \sum_{j \in S, j \neq i} \delta_j u_j.$$

However, if this remaining share is less than $\delta_i u_i$, then i will rather refuse to offer or make an offer which will be rejected by at least one member. In this case, agent i's expected payoff is $\delta_i u_i$. Overall, when S is chosen and i is selected to propose, i's payoff is

$$\max\{V(S) - \sum_{j \in S, j \neq i} \delta_j u_j, \delta_i u_i\}. \tag{3}$$

We introduce the following notation

$$z_S = \max\{V(S) - \sum_{k \in S} \delta_k u_k, 0\} \tag{4}$$

to indicate whether a coalition S will form. Namely, if $z_S > 0$ the gain of the coalition S is large enough, and S will form. This notation also simplifies the payoff formula in (3), in particular,

$$\max\{V(S) - \sum_{j \in S, j \neq i} \delta_j u_j, \delta_i u_i\} = \delta u_i + z_S.$$

Now, the probability of i being a proposer for coalition S is $\alpha_i \cdot P(S)$. Therefore, agent i's ex-ante payoff in any subgame is

$$u_i = \sum_{S \in \mathcal{F}: i \in S} \alpha_i \cdot P(S) \cdot (\delta_i u_i + z_S) + \left(1 - \sum_{S \in \mathcal{F}: i \in S} \alpha_i \cdot P(S)\right) \delta_i u_i.$$

This is equivalent to

$$u_i = \delta_i u_i + \sum_{S \in \mathcal{F}: i \in S} \alpha_i \cdot P(S) \cdot z_S,$$

and thus,

$$(1 - \delta_i) u_i = \alpha_i \cdot \sum_{S \in \mathcal{F}: i \in S} P(S) \cdot z_S. \tag{5}$$

We will show that $\vec{u}$ that satisfies the conditions above is the unique solution of the following optimization program [9]

$$\begin{aligned} \text{minimize: } & \sum_{i\in N} \frac{\delta_i(1-\delta_i)}{\alpha_i} u_i^2 + \sum_{S\in\mathcal{F}} P(S) z_S^2 \\ \text{subject to: } & \sum_{i\in S} \delta_i u_i + z_S \geq V(S) \quad \forall S \in \mathcal{F}, \\ & u_i \geq 0, z_S \geq 0 \;\; \forall i \in N, S \in \mathcal{F}. \end{aligned} \tag{A}$$

To prove the theorem, given a stationary equilibrium payoff u, let z be defined as in (4), and let

$$\lambda_S = 2P(S)z_S.$$

Consider λ_S as the dual variable corresponding to the constraint $\sum_{i\in S} \delta_i u_j + z_S \geq V(S)$ of (A). We now show x, z, λ satisfy both the first order and the complementary slackness conditions of (A), which shows that x, z is the unique solution of (A).

The first order conditions are

$$2\frac{\delta_i(1-\delta_i)}{\alpha_i} u_i = \sum_{S:i\in S} (\delta_i)\lambda_S \quad \text{and} \quad 2P(S)z_S = \lambda_S.$$

Both of these are satisfied because of the definition of λ_S and (5).

Furthermore, the complementary slackness condition also follows because:

- If $\sum_{i\in S} \delta_i u_i + z_S > V(S)$, then $z_S > V(S) - \sum_{i\in S} \delta_i u_i$, and because of (4), $z_S = 0$, which means the dual variable $\lambda_S = 0$.
- If $\lambda_S > 0$, then $z_S > 0$, therefore, $z_S = V(S) - \sum_{i\in S} \delta_i u_i$. This implies that the constraint S binds.

The proof of the second part of the theorem is standard. When a set S is chosen and i is the proposer, he will need to propose $\delta_j u_j$ to an agent $j \in S$ and i only does so if the remaining share is at least $\delta_i u_i$. Moreover, if $\sum_{j\in S} \delta_j u_j < V(S)$ then when $i \in S$ makes an offer, all other agents agree with probability 1, because otherwise, i

[9] Intuitively, equation (5) can be seen as a duality condition of an optimization problem, where z_S is the dual variable for a constraint S. However, z_S also appears in (4). Hence, we have a problem where z_S appears in both a primal and a dual program. To resolve this, for every $S \in \mathcal{F}$ we add a term $const_z \cdot z_S^2$ to the objective function of a convex program. The first order condition corresponding to the variable z_S captures a relation between z_S and the dual variable of the constraint $\sum_{i\in S} \delta_i u_i + z_S \geq V(S)$. By adjusting these constants in a right way, we obtain this program.

can offer $\delta_j u_j + \epsilon$ to j, with some small $\epsilon > 0$ to make all agents agree (in a sub-game perfect equilibrium) and gets a better payoff.[10] However, offering j $\delta_j u_j + \epsilon$ cannot be an equilibrium outcome. ∎

A.2 Proof of Theorem 2

In this proof, we denote the Euclidean norm of $\vec{u}$ by $||u||$. Recall that u^* is the optimal solution of (B) and $u^{*\delta}, z_S^\delta$ is the optimal solution of (A').

Consider $u_i = \frac{1}{\delta} u_i^*, z_S = 0$, it is straight forward to see that $(\vec{u}, \vec{z})$ is a feasible solution of (A'), for which $u^{*\delta}, z_S^\delta$ is the optimal solution. Therefore,

$$||\frac{u^*}{\delta}||^2 + 0 \geq \sum_{i \in N} (u_i^{*\delta})^2 + \frac{1}{\delta(1-\delta)} \sum_{S \in \mathcal{F}} P(S) \cdot (z_S^\delta)^2 \geq ||u^{*\delta}||^2,$$

which implies

$$||u^{*\delta}|| \leq \frac{1}{\delta} ||u^*||. \tag{6}$$

When δ tends to 1, z_S^δ approaches 0 because otherwise, the objective of (A') approaches ∞, which contradict to the fact that it is the optimal solution of (A'). Therefore, there exists $\epsilon(\delta)$ such that $\lim_{\delta \to 1} \epsilon(\delta) = 0$ and $z_S^\delta < \epsilon(\delta)$. Because of the constraints in (A'), we have

$$\sum_{i \in S} u_i^{*\delta} \geq V(S) - \epsilon(\delta) \ \ \forall S \in \mathcal{F}. \tag{7}$$

Note that u^* is the unique optimal solution of (B), thus, u^* is the intersection of two sets

$$\{\vec{u} : ||u|| \leq ||u^*||\} \text{ and } \{\vec{u} : \sum_{i \in S} u_i \geq V(S) \ \ \forall S\}.$$

Therefore, as δ approaches 1, the intersection of

$$\{\vec{u} : ||u|| \leq \frac{1}{\delta} ||u^*||\} \text{ and } \{\vec{u} : \sum_{i \in S} u_i \geq V(S) - \epsilon(\delta) \ \ \forall S\}$$

shrinks to the single point u^*. Because of (6) and (7), $u^{*\delta}$ is in this intersection. This shows that

$$u^{*\delta} \xrightarrow{\delta \to 1} u^*.$$

[10] Note that i sequentially makes offers to agents in S.

The second part of the theorem follows because $\lim_{\delta\to 1} u^{*\delta} = u^*$ and $\lim_{\delta\to 1} z^\delta = 0$. ■

A.3 Proof of Proposition 3.1

If the binding constraints are only the pairs in M, then deleting the constraints corresponding with $E - M$ will not affect the optimization program. When the constraints are only $u_i + u_j \geq V_{ij}$ for $(i, j) \in M$, the optimal solution is $u_i^* = u_j^* = V_{ij}/2$. ■

A.4 Proof of Proposition 3.4

According to Theorem 2 when $\delta \to 1$, the stationary equilibrium payoff of the bargaining game $\mathcal{B}(\delta, G, \{(s_k, b_k, V_k) : k \in K\})$ converges to the optimal solution of the following program

$$\begin{aligned} \text{minimize: } & \sum_{i\in N} u_i^2 \\ \text{subject to: } & \sum_{j\in P} u_j \geq V_k \quad \text{for all } P \text{ connecting } s_k, b_k. \end{aligned} \tag{8}$$

Due to Lemma 2.2, u^* is the optimal solution of (8) if and only if there exist $\lambda_P \geq 0$ such that

$$u_i^* = \sum_{P: i\in P} \lambda_P.$$

Furthermore, for a path P connecting s_k, b_k

$$\text{if } \sum_{i\in P} u_i^* > V_k, \text{ then } \lambda_P = 0 \text{ and if } \lambda_P > 0, \text{then } \sum_{i\in P} u_i^* = V_k.$$

This condition is exactly the condition for $\vec{\lambda}$ to be an equilibrium flow of $\mathcal{T}(G, \{(s_k, b_k, V_k) : k \in K\})$. Thus, the congestion for i coincides with u_i^*. ■

A.5 Proof of Proposition 3.5

Let u^* be the limit payoff of $u^{*\delta}$, then u^* is the optimal solution of

$$\begin{aligned} &\min \sum_{i\in N} u_i^2 \\ &\text{subject to } \sum_{j\in S} u_j \geq V(S) \quad \forall S \in \mathcal{F} \end{aligned} \tag{B}$$

To show that u^* is in the core, we only need to prove $\sum_{i\in N} u_i^* = V(N)$. Let S be the largest binding coalition. Using the supermodularity, we can show that if A, B are binding coalitions then $A \cap B$ and $A \cup B$ are also binding. Because of this observation, one can conclude that any agent outside S is not contained in any binding coalition.

From the convex program, we know that an agent that is not in any binding coalition gets 0 payoff. That is, $u_i^* = 0$ if $i \notin S$. Thus,

$$\sum_{i\in N} u_i^* = \sum_{i\in S} u_i^* = V(S) \leq V(N).$$

But $\sum_{i\in N} u_i^* \geq V(V)$ because of the constraint of the program, therefore, $\sum_{i\in N} u_i^* = V(N)$.

The proof of the second part is based on Theorem 2 and Lemma 2.1. We know that the program (B) characterizes the payoff at the limit $\delta \to 1$.

Let $S_1, S_2, \ldots, S_K$ be the set of sets whose corresponding constraints bind in this program. We first show that S_k are nested. Assume the contrary, let S, T be the two binding sets which are not nested. Let u^* be the solution of the optimization program above, then

$$V(S) + V(T) = \sum_{i\in S} u_i^* + \sum_{i\in T} u_i^* = \sum_{i\in S\cap T} u_i^* + \sum_{i\in S\cup T} u_i^*.$$

But because of the constraints in the optimization program,

$$\sum_{i\in S\cap T} u_i^* + \sum_{i\in S\cup T} u_i^* \geq V(S\cap T) + V(S\cup T).$$

These inequalities imply $V(S) + V(T) \geq V(S\cap T) + V(S\cup T)$, which contradicts to the strict supermodularity of V.

Now, $u^{*\delta}$ is the stationary equilibrium unique payoff of (N, V, δ). Thus, it is the optimal solution of A'. According to Lemma 2.1, there exists λ_S^δ such that

$$u_i^{*\delta} = \sum_{k:i\in S} \lambda_S^\delta,$$

and $\lambda_S^\delta = 0$ if S does not form in the game (N, V, δ).

Using Theorem 2, there exists δ_0 such that if $\delta > \delta_0$, then a set that is not one of S_k does not form at an equilibrium of (N, V, δ). Thus, $\lambda_S^\delta = 0$ if $S \notin \{S_1, \ldots, S_K\}$. This concludes the proof. ■

A.6 Proof of Corollary 3.7

Assume $u_i^{*\delta} < u_j^{*\delta}$, then because of Lemma 2.1, there is a binding coalition S that contains j but not i. Because i is in all the communities that j belongs to, thus $S' = S - \{j\} \cup \{i\}$ is also a feasible coalition. However,

$$\sum_{k\in S'} u_k^{*\delta} < \sum_{k\in S} u_k^{*\delta} = V(S) = V(S').$$

The last inequality holds because v only depends on the size of the coalition. But this is a contradiction to $\sum_{k\in S'} u_k^{*\delta} \geq V(S)$ because of the constraints of A'. ■

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