Calculating the nth voter utility under the infinite sequential voting with externalities model

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Abstract

We study a model of sequential voting with two alternatives, in which each voter has its own private preference, and some utility externalities. In this model, the voter prefers voting for the chosen winner over voting for his own private preference. Under this model, the voters act strategically as they're interested in maximizing their utility which depends on 3 factors: (i) the voter's preferred alternative, (ii) the voter's vote, and (iii) the outcome of the voting. In this model the voting goes on sequentially until the gap in votes between the two alternatives is at least some large value M. Alon-et-al show in [1] that the set of strategies in which every player plays the $k^* - threshold$ strategy form a subgame perfect equilibrium. In this work, under the assumption that each player plays the $k^* - threshold$ strategy, we offer a way to calculate the expected utility of each voter.

1 Introduction and Model

In this paper we examine the model of sequential voting suggested by Alon-et-al in [1], and offer a way to calculate the expected utility of each voter according to its place in the line of voters.

The model is depicted as follows: there are two alternatives, A and B, and a set of $S = \mathbb{N}$ of voters that are choosing one alternative as a winner. This means the population is infinite and countable. Each voter in S has a private preference (type) about the alternative he likes, either A or B. The private preferences of the different voters are drawn independently and identically from prior distribution, each voter prefers A with probability $p = \frac{1}{2}$ and prefers B with probability $q = \frac{1}{2}$.

In their model, the voters not only care about the chosen alternative but also about the alignment of their votes with the chosen alternative (conforming with the social choise). Thus, the utility of a voter depends on the alternative he likes, his vote, and the chosen alternative.

We set the utility as follows:

If a voter likes A, votes for A and A is chosen, the utility of the voter is α. The same holds for the symmetric case of a voter that likes B, votes for B, and B wins.

If a voter likes A, votes for B, and B wins, the utility of the voter is γ. The same holds for the symmetric case of a voter that likes B, votes for A, and A wins.

If a voter likes A, votes for A, and B wins, the utility of the voter is β. The same holds for the symmetric case.

If a voter likes A, votes for B, and A wins, the utility of the voter is δ. The same holds for the symmetric case.

We assume that $\alpha > \gamma > \max\{\beta, \delta\}$. This means that a voter obtains maximal utility if he votes for his preferred alternative and his preferred alternative wins. It also means that a voter prefers voting for the winner over voting for his preferred alternative.

In sequential voting, each voter observes the votes of all his predecessors before casting his publicly observed vote. Players move sequentially, one at a time (roll-call voting): player 1 moves first, player 2 moves second, and so on. In his move, the action of player i is to vote for either alternative A or alternative B. A strategy s_i of player $i \in S$, is a function from his type and the history of votes of players $1,2,3, \dots, i-1$ to an alternative, his own vote. Formally: $s_i: \{type\ A, type\ B\} \times \{A, B\}^{i-1} \rightarrow \{A, B\}$.

Since the population is infinite, to determine the end of the game we parameterize it by an integer $M > 0$, the "winning threshold". The game ends when one candidate leads by at least M votes over the other candidate. In this case the candidate with the larger number of votes is declared "the winner".

We only study the case where the two alternatives are a-priori symmetric, that is, the prior probability is $(\frac{1}{2}, \frac{1}{2})$. In [1] they define the $r - threshold$ strategy, in which, a player that likes A votes A if the number of votes for B minus the number of votes for A is at most r, otherwise it votes for B. Similarly, a player that likes B votes B if the number of votes for A minus the number of for B is at most r, otherwise it votes for A.

Let k^* be the largest integer such that $\frac{1}{k+1}\alpha + \frac{k}{k+1}\beta > \gamma$. In [1] they have proved the following theorem: The set of strategies in which every player plays the $k^* - threshold$ strategy form a subgame perfect equilibrium.

Notice that k^* does not depend on M, only on the utilities a, γ, β.

For the rest of this paper, we denote k^* to be the largest integer such that $\frac{1}{k+1}\alpha + \frac{k}{k+1}\beta > \gamma$ plus 1. With this k^*, the subgame perfect equilibrium strategy is the one in which all players play the $k^* - 1\ threshold$.

2 Example of calculating the utility of the first and second voter

In this section and the following section, we assume w.l.o.g that the nth player type is A, that is, that the nth player prefers alternative A over B. In order to make things easier, we'll always assume the number of votes have not reached k^* (or even assume $n < k^*$).

In order to calculate first player utility, we first introduce the following well-known fact about random walks, commonly termed as the "Gambler's Ruin Problem":

Consider an infinite sequence of i.i.d. random variables $x_1, x_2, \dots$ where

$$Pr(x_i = 1) = Pr(x_i = -1) = \frac{1}{2}.$$

Let $S_n = \sum_{i=1}^{n} x_i$, and $T = \min\{n: S_n = a \text{ or } S_n = -b\}$ for two nonnegative integers a, b. Then

$$Pr(S_T = a) = \frac{b}{a+b}$$

And

$$\Pr(\mathrm{S_T} = -b) = \frac{a}{a+b}$$

In other words, if an infinite random walk starts at 0 and ends when it either reaches a or $-b$, then the probability it ends at a is $\frac{b}{a+b}$ and the probability it ends at $-b$ is $\frac{a}{a+b}$.

According to the $k^* - threshold$ strategy, first player votes A. All other players vote for A with a-priori probability of $\frac{1}{2}$ and vote for B with a-priori probability of $\frac{1}{2}$.

We'll calculate the player's expected utility using the "Gambler's Ruin Problem" solution.

After the first player casts his vote, alternative A leads by 1 vote, therefore:

$$\Pr(A\ wins) = \frac{k^* + 1}{2k^*}$$

$$\Pr(B\ wins) = \frac{k^* - 1}{2k^*}$$

Therefore the expected utility of the first player is $\frac{k^*+1}{2k^*}\alpha + \frac{k^*-1}{2k^*}\beta = \frac{(k^*+1)\alpha+(k^*-1)\beta}{2k^*}$.

Next we calculate the expected utility of the second player. We'll assume for now that $k^* > 1$ (therefore the second player always votes for his preference). Again assume w.l.o.g that the second player prefers alternative A over B.

With probability $\frac{1}{2}$ preference A leads by 1 vote, and the expected utility of the second player in this case is:

$$\frac{k^* + 2}{2k^*}\alpha + \frac{k^* - 2}{2k^*}\beta = \frac{(k^* + 2)\alpha + (k^* - 2)\beta}{2k^*}$$

With probability $\frac{1}{2}$ preference B leads by 1 vote, and the expected utility of the second player in this case is:

$$\frac{k^*}{2k^*}\alpha + \frac{k^*}{2k^*}\beta = \frac{k^*\alpha + k^*\beta}{2k^*}$$

Therefore the expected utility of the second player is:

$$\frac{(2k^* + 2)\alpha + (2k^* - 2)\beta}{4k^*} = \frac{(k^* + 1)\alpha + (k^* - 1)\beta}{2k^*}$$

This calculation was an example for when we assume that we can never reach k^* vote gap during the $n - 1$ votes.

Things get more complicated without this assumption, which we need to remove in order to calculate the different utilities of the players.

We'll denote by $Votes(A_n)$ and $Votes(B_n)$ the number of votes for A and B after n votes (this means that $Votes(A_n) + Votes(B_n) = n$). We'll also denote the difference $X_n = Votes(A_n) - Votes(B_n)$. For the n-th player, when coming to cast his vote, there are 3 options, either the threshold has been reached in any of the previous votes, which means that for some $m < n$, $|X_m| = k$, and then all players after the $m - th$ player voted for the leading alternative, and player n, following the k-threshold strategy, also votes for the leading alternative and wins a utility of γ. (Note that we don't consider the case where $X_{n-1} = M$, which means that the voting was already over when it's the n-th player turn to vote, we assume that the n-th player does vote in order to calculate his utility).

3 Gambler's Ruin in Finite Time

This section goes over the gambler's ruin problem in finite time, and presents a solution to the problem as shown in [2] using lattices.

The gambler's ruin problem as described in the previous section assumes infinite time (which actually means infinite moves or infinite votes). In order to calculate the utility of player (voter) n in our case we first calculate the probability $\Pr(|X_{n-1}| = c)$ for every integer $c \leq k^*$ and the probability $\Pr(|X_{n-1}| \geq k^*)$.

The Gambler's Ruin Problem In Finite Time: Suppose a gambler starts with j dollars, and in each wager has a change of p to earn 1 dollar, and a chance of q to lose one dollar. The probability of having k dollars after making n bets, is denoted by $P_{j,k}^n$ for $k = 0,1,2,3 \dots H$. If at some stage the gambler has either 0 dollars or H dollars, the gambling stops (the gambler has either lost all of his money or reached his goal of H dollars).

By the theory of Markov Chains, denote

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ q & 0 & p & 0 & & 0 \\ 0 & q & 0 & p & 0 & \\ & & \ddots & & \ddots & \\ 0 & 0 & 0 & 0 & & 1 \end{pmatrix}$$

Then P^n has entries $P^n_{j,k}$. Note that $P^n_{j,0}$ and $P^n_{j,H}$ are the probabilities of being absorbed by time n. That is, the probability that by time n the player has no money left is $P^n_{j,0}$, and $P^n_{j,H}$ is the probability of meeting his goal of H dollars by time n.

Reference [2] presents a solution to the Gamblers Ruin Problem In Finite Time using lattices, based on the following fact:

For $1 \le j, k \le H$, let $L^n_{j,k}(H)$ represent the collection of all lattice paths going from j to k in n-steps, bounded by horizontal lines $y = 0$ and $y = H$ and restricted not to hit these boundaries.

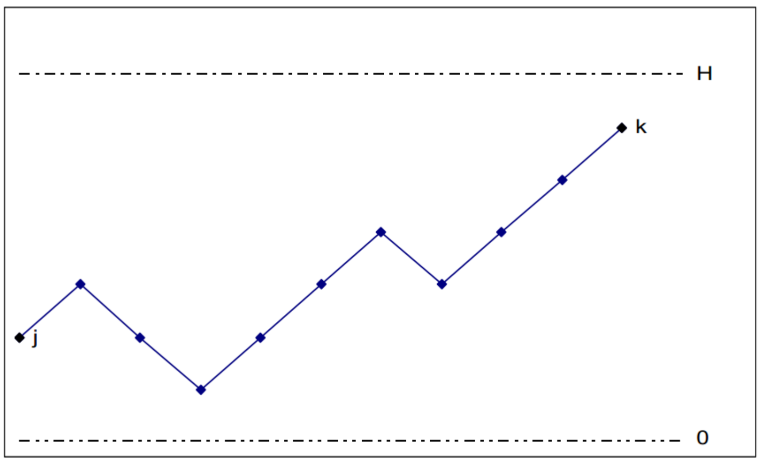

Figure 1 - Taken from [2]

To determine $P^n_{j,k}$, the problem comes down to finding the number of lattice paths going from j to k in n-steps not hitting zero or H. The number of lattice paths in $L^n_{j,k}(H)$ is represented by $|L^n_{j,k}(H)|$.

Using the proof in [3, 4], [2] we can state the following corollary:

Corollary 1

Case 1 $1 \leq j, k \leq H$

$$P_{j,k}^{n} = \frac{2}{H} p^{\frac{k-j}{2}} q^{\frac{j-k}{2}} \sum_{u=1}^{H} \sin\left(\frac{u\pi k}{H}\right) \sin\left(\frac{u\pi j}{H}\right) \left[2\sqrt{pq}\cos\left(\frac{u\pi}{H}\right)\right]^{n}$$

Case 2 $1 \leq j \leq H-1$

$$P_{j,0}^{n} = \frac{2}{H} p^{\frac{1-j}{2}} q^{\frac{1+j}{2}} \sum_{u=1}^{H} \left[2\sqrt{pq}\cos\left(\frac{u\pi}{H}\right) - 1\right]^{-1} \sin\left(\frac{u\pi j}{H}\right) \sin\left(\frac{u\pi}{H}\right) [\left(2\sqrt{pq}\cos\left(\frac{u\pi}{H}\right)\right)^{n} - 1]$$

4 Computing the utility of the nth voter

We look at two different cases, first where $|X_{n-1}| < k^*$, second where $|X_{n-1}| \geq k^*$.

First case: $|X_{n-1}| < k^*$

First notice that for $-k^* < c < k^*$, $P(X_{n-1} = c) = P_{k^*,k^*+c}^{n-1}$ assuming $H = 2k^*$.

Explanation: X_{n-1} indicates the vote difference between alternative A and alternative B after $n-1$ votes (i.e. $Votes(A_{n-1}) - Votes(B_{n-1})$). Under our assumption that all players play the $k^* - threshold$ strategy, this situation occurs only if the vote difference has never exceeded k^* or declined below – k^* (that is, for every $b < n-1$, $|X_b| < k^*$). Thus, getting to a vote difference of c the is the same as not exceeding $2k^*$, or declining below 0, when starting with a vote difference of k^*, and getting a vote difference of $k^* + c$, which is exactly P_{k^*,k^*+c}^{n-1} (with $H = 2k^*$).

Assuming the vote difference is c after $n-1$ votes, we can use the solution for the Gambler's Ruin Game (with infinite voting of course) to calculate the probability of hitting a vote difference of either k^* before – k^* with infinite voting (which according to [1], equal to either alternative A winning or alternative B winning).

$$P(hitting\ a\ vote\ difference\ of\ k^*\ before\ -k^* |\ vote\ difference\ c) =$$

$$\frac{k^* + c}{(k^* - c) + (k^* + c)} = \frac{k^* + c}{2k^*}$$

Similarly,

$$P(hitting\ a\ vote\ difference\ of -k^*\ before\ k^* |\ vote\ difference\ c) =$$

$$\frac{k^* - c}{(k^* - c) + (k^* + c)} = \frac{k^* - c}{2k^*}$$

In this case, assuming the nth player prefers alternative A, his expected utility is

$$P(k^*\ difference\ is\ achieved\ first\ |\ c + 1\ votes) \cdot \alpha +$$

$$P(-k^*\ difference\ is\ achieved\ first\ |\ c + 1\ votes) \cdot \beta +$$

$$\frac{k^* + (c+1)}{2k^*} \cdot \alpha + \frac{k^* - (c+1)}{2k^*} \cdot \beta = \frac{k^*(\alpha + \beta) + (c+1)(\alpha - \beta)}{2k^*}$$

Second case: $|X_{n-1}| \geq k^*$

Similar to the first case, $P(X_{n-1} \geq k^*) = P^{n-1}_{k^*,2k^*} = P^{n-1}_{k^*,0} = P(X_{n-1} \leq -k^*)$ assuming $H = 2k^*$ (The second equality is easy to explain by symmetry).

In the case that $|X_{n-1}| \geq k^*$, player n votes according the $k^* - threshold\ strategy$, and its utility is easy to calculate. With probability $\frac{1}{2}$, the utility is α (if $X_{n-1} > k^*$), and with probability $\frac{1}{2}$, the utility is γ (in the case that $X_{n-1} < k^*$), assuming that the nth player assumes alternative A (This calculation is true both for the case where the nth player prefers alternative A, and for the case where the nth player prefers alternative B, because we are averaging the expected value because of symmetry).

As a result, giving that we are in the second case, the nth player utility is $\frac{\alpha+\gamma}{2}$.

Using the law of total expectation, we can now calculate the utility of the nth voter:

$$E[utility\ of\ the\ nth\ voter] = P^{n-1}_{k^*,0} \cdot \alpha + P^{n-1}_{k^*,2k^*} \cdot \gamma + \sum_{c=1}^{n-1} P^{n-1}_{k^*,k^*+c} \cdot \frac{k^*(\alpha + \beta) + c(\alpha - \beta)}{4k^*}$$

Using **Corollary 1**, placing $p = q = \frac{1}{2}$, $H = 2k^*$, we get

Theorem 1

$$E_n = E[utility\ of\ the\ nth\ voter] =$$

$$(\alpha+\gamma)\frac{1}{2k^*}\sum_{i=0}^{n-1-1}\left(\frac{1}{2}\right)^i\sum_{u=1}^{2k^*}\sin\left(\frac{u\pi}{2k^*}\right)\sin\left(\frac{u\pi}{2}\right)\left[2\cos\left(\frac{u\pi}{2k^*}\right)\right]^i +$$

$$\frac{1}{k^*}\sum_{c=-k^*+1}^{k^*-1}\frac{k^*(\alpha+\beta)+(c+1)(\alpha-\beta)}{2k^*}\sum_{u=1}^{2k^*}\sin\left(\frac{u\pi(k^*+c)}{2k^*}\right)\sin\left(\frac{u\pi}{2}\right)\left[\cos\left(\frac{u\pi}{2k^*}\right)\right]^{n-1}$$

Remember that using alon-et-al [1], k^* is the largest integer such that $\frac{1}{k+1}\alpha + \frac{k}{k+1}\beta > \gamma$, or with simple math, the largest integer such that $k < \frac{\alpha-\gamma}{\gamma-\beta}$. As stated before, we are using $k^* + 1$ instead of k^* to simplify the formula.

5 Taking a go at simplifying the formula for the utility

$$E_n = E[utility\ of\ the\ nth\ voter] =$$

$$(\alpha+\gamma)\frac{1}{2k^*}\sum_{i=0}^{n-1-1}\left(\frac{1}{2}\right)^i\sum_{u=1}^{2k^*}\sin\left(\frac{u\pi}{2k^*}\right)\sin\left(\frac{u\pi}{2}\right)\left[2\cos\left(\frac{u\pi}{2k^*}\right)\right]^i +$$

$$\frac{1}{k^*}\sum_{c=-k^*+1}^{k^*-1}\frac{k^*(\alpha+\beta)+(c+1)(\alpha-\beta)}{2k^*}\sum_{u=1}^{2k^*}\sin\left(\frac{u\pi(k^*+c)}{2k^*}\right)\sin\left(\frac{u\pi}{2}\right)\left[\cos\left(\frac{u\pi}{2k^*}\right)\right]^{n-1}$$

For all even u $sin(u * \frac{\pi}{2}) = 0$

For all $u = 1,5,9 \ldots sin(\frac{u\pi}{2} + \frac{cu\pi}{2k^*})sin(\frac{u\pi}{2}) = cos(\frac{cu\pi}{2k^*}) \cdot 1$

For all $u = 3,7,11 \ldots\ sin(\frac{u\pi}{2} + \frac{cu\pi}{2k^*})sin(\frac{u\pi}{2}) = -cos(\frac{cu\pi}{2k^*}) \cdot (-1)$

$$E_n = E[utility\ of\ the\ nth\ voter] =$$

$$(\alpha+\gamma)\frac{1}{2k^*}\sum_{i=0}^{n-1-1}\left(\frac{1}{2}\right)^i\sum_{u=1}^{2k^*}\sin\left(\frac{u\pi}{2k^*}\right)\sin\left(\frac{u\pi}{2}\right)\left[2\cos\left(\frac{u\pi}{2k^*}\right)\right]^i +$$

$$\frac{1}{k^*}\sum_{c=-k^*+1}^{k^*-1}\frac{k^*(\alpha+\beta)+(c+1)(\alpha-\beta)}{2k^*}\sum_{u=1,3,5\ldots}^{2k^*}\cos\left(\frac{u\pi c}{2k^*}\right)\left[\cos\left(\frac{u\pi}{2k^*}\right)\right]^{n-1}$$

If we look at $\sum_{c=-k^*+1}^{k^*-1}\frac{(c)(\alpha-\beta)}{2k^*}\sum_{u=1,3,5\ldots}^{2k^*}\cos\left(\frac{u\pi c}{2k^*}\right)\left[\cos\left(\frac{u\pi}{2k^*}\right)\right]^{n-1}$, we can see this term zeros out as $cos(a) = cos(-a)$.

$$E_n = E[utility\ of\ the\ nth\ voter] =$$

$$(\alpha+\gamma)\frac{1}{2k^*}\sum_{i=0}^{n-1-1}\left(\frac{1}{2}\right)^i\sum_{u=1}^{2k^*}\sin\left(\frac{u\pi}{2k^*}\right)\sin\left(\frac{u\pi}{2}\right)\left[2\cos\left(\frac{u\pi}{2k^*}\right)\right]^i +$$

$$\frac{1}{k^*}\sum_{c=-k^*+1}^{k^*-1}\frac{k^*(\alpha+\beta)+(\alpha-\beta)}{2k^*}\sum_{u=1,3,5\ldots}^{2k^*}\cos\left(\frac{u\pi c}{2k^*}\right)\left[\cos\left(\frac{u\pi}{2k^*}\right)\right]^{n-1}$$

$$E_n = E[utility\ of\ the\ nth\ voter] =$$

$$(\alpha+\gamma)\frac{1}{2k^*}\sum_{i=0}^{n-1-1}\left(\frac{1}{2}\right)^i\sum_{u=1}^{2k^*}\sin\left(\frac{u\pi}{2k^*}\right)\sin\left(\frac{u\pi}{2}\right)\left[2\cos\left(\frac{u\pi}{2k^*}\right)\right]^i +$$

$$\frac{1}{k^*}\frac{k^*(\alpha+\beta)+(\alpha-\beta)}{2k^*}\sum_{u=1,3,5\ldots}^{2k^*}\left[\cos\left(\frac{u\pi}{2k^*}\right)\right]^{n-1}\sum_{c=-k^*+1}^{k^*-1}\cos\left(\frac{u\pi c}{2k^*}\right)$$

$$E_n = E[utility\ of\ the\ nth\ voter] =$$

$$(\alpha+\gamma)\frac{1}{2k^*}\sum_{i=0}^{n-1-1}\left(\frac{1}{2}\right)^i\sum_{u=1}^{2k^*}\sin\left(\frac{u\pi}{2k^*}\right)\sin\left(\frac{u\pi}{2}\right)\left[2\cos\left(\frac{u\pi}{2k^*}\right)\right]^i +$$

$$\frac{1}{k^*}\frac{k^*(\alpha+\beta)+(\alpha-\beta)}{2k^*}\sum_{u=1,3,5\ldots}^{2k^*}\left[\cos\left(\frac{u\pi}{2k^*}\right)\right]^{n-1}(1+2)\sum_{c=1}^{k^*-1}\cos\left(\frac{\mathrm{u\pi c}}{\mathrm{2k^*}}\right)$$

What we have here is a dirchlet kernel, using $c = k, n = k^* - 1, x = \frac{u\pi}{2k^*}$

(based on http://en.wikipedia.org/wiki/Dirichlet_kernel)

$$E_n = E[utility\ of\ the\ nth\ voter] =$$

$$(\alpha+\gamma)\frac{1}{2k^*}\sum_{i=0}^{n-1-1}\left(\frac{1}{2}\right)^i\sum_{u=1}^{2k^*}\sin\left(\frac{u\pi}{2k^*}\right)\sin\left(\frac{u\pi}{2}\right)\left[2\cos\left(\frac{u\pi}{2k^*}\right)\right]^i +$$

$$\frac{1}{k^*}\frac{k^*(\alpha+\beta)+(\alpha-\beta)}{2k^*}\sum_{u=1,3,5\ldots}^{2k^*}\frac{\left[\cos\left(\frac{u\pi}{2k^*}\right)\right]^{n-1}\sin\left(\frac{\left(k^*-\frac{1}{2}\right)u\pi}{2k^*}\right)}{\sin\left(\frac{u\pi}{4k^*}\right)}$$

6 Open questions and further work

- Can we find a simpler formula (maybe one that is more numerically stable) to calculate the utility of the nth voter?
- Can we put a limit on the ratio between the utility of the best placed player and the utility of the worst placed player?
- What happens if we change the model so that the first player to get some percentage of the votes wins (instead of a constant amount M)? – this will also change k^* (most likely to non-constant value as well).
- In the above calculation, we always assume the nth player does vote, what if this is not the case? (What if we take into account the probability that the nth player does not vote because the voting has ended already). What if we consider some utility for not-voting at all (somewhere between γ and β?
- What happens when n goes to infinity in our model? What happens when n goes to infinity in the proposed models above (winning is declared when some alternative is leading by some percentage P, or when non-voting have some utility).
- My conjecture (not proven): E_n is monotonous decreasing. This conjecture can help in answering question 1 by dividing the first player by the nth player utility as n goes to infinity.

References:

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