The Price of Anarchy in Network Creation Games Is (Mostly) Constant

Matúš Mihalák and Jan Christoph Schlegel

Institute of Theoretical Computer Science, ETH Zurich, Switzerland

Abstract. We study the price of anarchy and the structure of equilibria in network creation games. A network creation game (first defined and studied by Fabrikant et al. [4]) is played by n players $\{1, 2, \dots, n\}$, each identified with a vertex of a graph (network), where the strategy of player i, $i = 1, \dots, n$, is to build some edges adjacent to i. The cost of building an edge is $\alpha > 0$, a fixed parameter of the game. The goal of every player is to minimize its *creation cost* plus its *usage cost*. The creation cost of player i is α times the number of built edges. In the SumGame (the original variant of Fabrikant et al. [4]) the usage cost of player i is the sum of distances from i to every node of the resulting graph. In the MaxGame (variant defined and studied by Demaine et al. [3]) the usage cost is the eccentricity of i in the resulting graph of the game. In this paper we improve previously known bounds on the price of anarchy of the game (of both variants) for various ranges of α, and give new insights into the structure of equilibria for various values of α. The two main results of the paper show that for $\alpha > 273 \cdot n$ all equilibria in SumGame are trees and thus the price of anarchy is constant, and that for $\alpha > 129$ all equilibria in MaxGame are trees and the price of anarchy is constant. For SumGame this (almost) answers one of the basic open problems in the field – is price of anarchy of the network creation game constant for all values of α? – in an affirmative way, up to a tiny range of α.

1 Introduction

Network creation game, as defined and introduced by Fabrikant et al. in [4], is a game that models the process of building large autonomous computer and communication networks (such as the Internet). In this game, as in the reality, these networks are built and maintained by entities (*players* in the game-theoretic jargon) that pursue their own goals that may be different from the goals of other players – the players do not necessarily cooperate, they are *selfish* (we leave the meaning of this on an intuitive level). Network creation games is a well-studied and well-known research topic which is covered by many lectures and courses on algorithmic game theory and related subjects.

Network creation game is a strategic game with n players where each player is identified with a vertex (of a to be built graph/network). Every player i has to decide what edges incident to i the player *creates* (or *buys*, or *builds*). Building one edge costs the player $\alpha > 0$, which is a fixed parameter of the game. The

S. Kontogiannis, E. Koutsoupias, P.G. Spirakis (Eds.): SAGT 2010, LNCS 6386, pp. 276–287, 2010.

edges that the players buy form a graph (network) which is the result of the game. The players pursue two incompatible goals: pay as little as possible (minimize the *creation cost*), and have a good connection to other nodes of the network (maximize the *usage utility*). The usage utility of player i has been originally expressed as the following usage cost: the sum of distances to all other players in the resulting network [4] (where naturally players want to minimize this sum). Recently, the game where the usage cost of player i is expressed as the maximum distance of i to any node of G has been studied [3]. In this paper we consider both variants.

The central question that motivated the study of network creation games is: what do we lose in terms of quality of a network, if the communication network is built autonomously by selfish agents, as opposed to a communication network that is centrally planned and built? The *price of anarchy* of a game is a way to express this in that one compares the cost of a worst Nash equilibrium[1] (worst in the sense of the cost of the network) with the cost of an optimum network – the ratio of these two values is the *price of anarchy* of the game.

Definition of the game and related concepts. Let $G = (V, E)$ be an undirected graph (and we shall only consider undirected graphs in the following). For $u, v \in V$ we denote by $d_G(u, v)$ the length of a shortest u-v-path in G, and by $D_G(v)$ the *eccentricity* of the vertex v, i.e., the maximum distance between v and any other vertex of G. If G is not connected we define $d_G(u, v) := \infty$. We denote the degree of vertex $v \in V$ in G by $\deg_G(v)$. The *average degree* of G is $\deg(G) := \frac{1}{|V|}\sum_{v\in V} \deg_G(v) = \frac{2|E|}{|V|}$. We sometimes omit the index G and write simply $d(u, v)$, $D(v)$, or $\deg(v)$ if the underlying graph G is clear from the context. For $k \in \mathbb{N}$ we define the *k-neighborhood* of a vertex $v \in V$ as the set $N_k(v) := \{w \in V : d(v, w) \leq k\}$ (observe that v belongs to N_k), and the *boundary of the k-neighborhood* as the set $N_k^{=}(v) := \{w \in V : d(v, w) = k\}$. Furthermore we define the set of all eccentric vertices of v by $\mathcal{E}(v) := N_{D(v)}^{=}(v)$. We denote the *diameter* of G by $\text{diam}(G)$, and the *radius* of G by rad(G). Recall that $\text{diam}(G) = \max_{u,v\in V} d(u, v)$ and $\text{rad}(G) = \min_v D(v)$. A *central vertex* is a vertex v for which $D(v) = \text{rad}(G)$. Graph G is a *star* if it is a tree and all edges of G are incident to one vertex. Recall that a *biconnected graph* is a graph that does not contain a *cut vertex*, i.e. a vertex whose removal makes the graph disconnected, and that a *biconnected component* (or *block*) of a graph G is a maximal biconnected subgraph of G.

We consider n *players* $N = \{1, \ldots, n\}$ in our setting. Let $\alpha > 0$ be a real number which we shall call the *edge price*. The set of *strategies* of player $i \in N$ is the set $S_i = 2^{N\setminus\{i\}}$ (i.e., S_i contains all subsets of the set $N \setminus \{i\}$). A strategy $s_i \in S_i$ corresponds to a set of players to which i *buys* (or *builds*) an edge. We define $S := S_1 \times S_2 \times \cdots \times S_n$ and call the elements of S the *strategy profiles*. For every strategy profile $s \in S$ we define the graph $G(s) := (N, \bigcup_{i=1}^{n} \bigcup_{j\in s_i} \{\{i, j\}\})$, and a cost function $c_i(s)$ for every player i (to be specified later). The triple (N, S, c), where $c : S \longrightarrow \mathbb{R}^n$ is given by $c(s) := (c_1(s), \ldots c_n(s))$, naturally

[1] In Nash equilibrium no player can unilaterally change its strategy and improve.

defines a non-cooperative n-player strategic game. Depending on the form of $c_i(\cdot)$ we distinguish two games. *Sum-Unilateral Network Creation Game*, or shortly SumGame, is the game given by (N, S, c) where for $s \in S$, $i \in N$,

$$c_i(s) = \alpha \cdot |s_i| + \sum_{j=1,\ldots,n} d_{G(s)}(i,j).$$

Max-Unilateral Network Creation Game, or shortly MaxGame, is the game given by (N, S, c) where for $s \in S$, $i \in N$,

$$c_i(s) = \alpha \cdot |s_i| + \max_{j=1,\ldots,n} d_{G(s)}(i,j).$$

We call the term $\alpha \cdot |s_i|$ in both cost functions the *creation cost*, and the term $\sum_{j=1,\ldots,n} d_{G(s)}(i,j)$ or $\max_{j=1,\ldots,n} d_{G(s)}(i,j)$ in the respective cost function the *usage cost* of player i. A *Nash equilibrium* (NE for short) of the game (N, S, c) is a strategy-profile $s \in S$ such that for every player $i \in N$ and every strategy $\tilde{s}_i \in S_i$ we have $c_i(s) \leq c_i(s_1, \ldots, s_{i-1}, \tilde{s}_i, s_{i+1}, \ldots, s_n)$, i.e. no player can lower her cost by changing her strategy when all other players keep their strategies unchanged. Observe therefore that for every finite α every NE is a connected graph, and in every NE any edge is bought by at most one player. If $s \in S$ is a Nash equilibrium of the game (N, S, c), we call $G(s)$ an *equilibrium graph* or sometimes a *stable graph.* The *social cost* C of a strategy-profile $s \in S$ is defined, for the respective cost function of player i, as the sum of the individual costs of the players under this strategy-profile, i.e.:

$$C(s) = \sum_{i=1}^{n} c_i(s) = \begin{cases} \alpha \cdot \sum_{i=1}^{n} |s_i| + \sum_{i=1}^{n} \sum_{j=1}^{n} d_{G(s)}(i,j) & \text{in SumGame,} \\ \alpha \cdot \sum_{i=1}^{n} |s_i| + \sum_{i=1}^{n} \max_{j=1,\ldots,n} d_{G(s)}(i,j) & \text{in MaxGame.} \end{cases}$$

Since for every graph $G = (V, E)$ on n vertices there is a strategy-profile inducing this graph, the social cost function generalizes for any graph G on n vertices: $C(G) = \alpha \cdot |E| + \sum_{v \in V} D_G(v)$, or $C(G) = \alpha \cdot |E| + \sum_{v \in V} \sum_{w \in V} d_G(v,w)$ for the respective cost function. We call a graph G_{OPT} minimizing the respective social cost function a *social optimum.* The *price of anarchy* (PoA for short) of a game (SumGame or MaxGame) is defined as $\max_{s \in S; s \text{ is NE}} \frac{C(G(s))}{C(G_{\text{OPT}})}$.

Related work. Networks have been an important research topic in the economical and social sciences, as networks naturally model relationships between interacting entities. As such, a link between two entities is usually created upon mutual consensus ("if entity A knows entity B then entity B knows entity A" is a common assumption). For an overview of economical and social studies from this perspective we refer to the book by Jackson [5] and to the references therein. Strategic network formation in this framework has been studied with tools from cooperative game theory. The trade-off between efficiency and stability for these kind of networks has been studied in Jackson and Wolinski [6].

We study networks where links can be created unilaterally (i.e., without an explicit agreement of both players at the ends of the respective edges) and where the payoff of the players reflects the cost for building the edges as well as the quality of the resulting network in terms of the players' distances in the network. The first game of this nature studied in the literature is SumGame.

Fabrikant et al. introduced and defined SumGame in [4]. They proved an upper bound $\mathcal{O}(\sqrt{\alpha})$ on PoA (by showing that PoA is bounded by the diameter of the equilibrium graph), and showed that every NE which is a tree has constant PoA (we will use this result later on). Albers et al. [1] showed that PoA is constant for $\alpha = \mathcal{O}(\sqrt{n})$ (this was also independently and earlier discovered by Lin [7]) and for $\alpha \geq 12n \lg n$. The latter result is achieved by showing that for $\alpha \geq 12n \lg n$ all NE are trees. Albers et al. also show the general bound $15(1 + (\min\{\alpha^2/n, n^2/\alpha\})^{1/3})$ for all α, which shows that PoA is $O(n^{1/3})$ for all α. Demaine et al. [3] show that PoA is constant already for $\alpha = \mathcal{O}(n^{1-\varepsilon})$ for any fixed ε, and show the general bound $2^{O(\sqrt{\lg n})}$ on PoA for all α.

Demaine et al. introduced and defined MaxGame in [3] as a natural variant of the network creation games, and showed that PoA is at most 2 for $\alpha \geq n$, $\mathcal{O}(\min\{4^{\sqrt{\lg n}}, (n/\alpha)^{1/3}\})$ for $2\sqrt{\lg n} \leq \alpha \leq n$, and $\mathcal{O}(n^{2/\alpha})$ for $\alpha < 2\sqrt{\lg n}$.

Recently, a related model has been introduced by Alon et al. in [2] where players do not buy edges, but only swap the endpoints of existing edges. Alon et al. claim some implications of their model to the models studied in this paper; this, however, seems not to be the case in the claimed extent (but it is not easy to argue as Alon et al. do not state any such claim formally). We refer to [8], the full version of this paper, for a detailed discussion.

Our results. For MaxGame we show that PoA is constant for $\alpha > 129$ and $\alpha = \mathcal{O}(n^{-1/2})$, and also prove that PoA is $2^{\mathcal{O}(\sqrt{\log n})}$ for any $\alpha > 0$ in Section 2. The result for $\alpha > 129$ is obtained as a corollary of the more general result (proved in Section 2.1) showing that in MaxGame for $\alpha > 129$ all equilibrium graphs are trees. This is proved by new techniques which establish and use estimates on the average degree of biconnected components of equilibrium graphs. In Section 3 we adopt the new techniques for SumGame to prove that for $\alpha > 273n$ all equilibrium graphs are trees. This result implies a constant upper bound on PoA for $\alpha > 273n$ which shrinks the range of edge-prices for which we do not know a constant upper bound to $\alpha = \Theta(n)$. A comparison and overview of the previously known bounds and the new bounds on PoA in both game variants are summarized in Table 1 and Table 2.

2 Bounding the Price of Anarchy in MaxGame

In this section we consider MaxGame. First we classify social optima. This is rather a folklore and resembles in many aspects the previously shown characterization of social optima in SumGame. We use this to bound PoA in MaxGame for small values of α.

Proposition 1. *For $\alpha \leq \frac{2}{n-2}$ the complete graph is a social optimum. For $\alpha \geq \frac{2}{n-2}$ the star is a social optimum.*

Table 1. Comparison of the previously known bounds for the price of anarchy in MaxGame (due to [3]) and the bounds proved in this paper. The abbreviations T. and C. stand for Theorem and Corollary, respectively.

α	0 – $\frac{1}{n-2}$	$\frac{1}{n-2}$ – $\mathcal{O}(n^{-\frac{1}{2}})$	$\mathcal{O}(n^{-\frac{1}{2}})$ – 129	129 – $2\sqrt{\log n}$	$2\sqrt{\log n}$ – n	n – ∞
new	1 (T. 1)	$\Theta(1)$ (C. 2)	$2^{\mathcal{O}(\sqrt{\log n})}$ (T. 3)	< 4 (C. 4)		≤ 2
old	$\mathcal{O}(n^{2/\alpha})$				$\mathcal{O}(\min\{4^{\sqrt{\log n}}, (n/\alpha)^{1/3}\})$	≤ 2

Table 2. Summary of the best known bounds for the price of anarchy in SumGame

α	0 – 1	1 – 2	2 – $\sqrt[3]{n/2}$	$\sqrt[3]{n/2}$ – $\sqrt{n/2}$	$\sqrt{n/2}$ – $O(n^{1-\epsilon})$	$O(n^{1-\epsilon})$ – $273n$	$273n$ – $12n\lg n$	$12n\lg n$ – ∞
PoA	1	$\leq \frac{4}{3}$ ([4])	≤ 4 ([3])	≤ 6 ([3])	$\Theta(1)$ ([3])	$2^{\mathcal{O}(\sqrt{\log n})}$ ([3])	< 5 (T. 6)	≤ 1.5 ([1])

Theorem 1. *For* $\alpha < \frac{1}{n-2}$ *the price of anarchy is* 1*. For* $\alpha < \frac{2}{n-2}$ *the price of anarchy is at most* 2*.*

The proofs of the two statements can be found in the full version of this paper [8]. Next we relate the diameter of an equilibrium graph with PoA of the game, where the following lemma is the key ingredient. The lemma exploits that a breadth-first search tree (BFS-tree) of an equilibrium graph already contains much information about the whole graph. For SumGame a similar result with a similar proof is known [1].

Lemma 1. *If* $G = (V, E)$ *is an equilibrium graph then* $C(G) \leq (2\alpha + 1)(n - 1) + n \cdot \mathrm{rad}(G)$*.*

Proof. Let T be a BFS-tree of G rooted in a central vertex v_0 of G. Let $v \in V \setminus \{v_0\}$. Let E_v be the edges built by v in T. Consider the following strategy of v: Buy all edges of E_v plus buy the edge to v_0. The creation cost of v in this strategy is at most $\alpha(|E_v| + 1)$ and the usage cost is at most $D(v_0) + 1$. As G is an equilibrium, every vertex (player) achieves in G the best possible cost, given what other players do. Thus, the above mentioned strategy upper-bounds the cost of v in equilibrium, i.e., $c_v(G) \leq \alpha(|E_v| + 1) + \mathrm{rad}(G) + 1$. For vertex v_0 we have $c_{v_0}(G) = \alpha|E_{v_0}| + \mathrm{rad}(G)$. Summing the obtained inequalities for every vertex of G yields the claimed inequality. □

Corollary 1. *Let* G *be a worst NE for* $\alpha \geq \frac{2}{n-2}$*. The price of anarchy is* $\mathcal{O}\left(1 + \frac{\mathrm{diam}(G)}{\alpha+1}\right)$*.*

Proof. By Proposition 1 and Lemma 1 we get

$$\mathrm{PoA} \leq \frac{(2\alpha+1)(n-1) + n \cdot \mathrm{rad}(G)}{(\alpha+2)(n-1)+1} \leq \frac{2\alpha+1}{\alpha+2} + \frac{n \cdot \mathrm{rad}(G)}{(n-1)(\alpha+2)} \leq 2 + \frac{2 \cdot \mathrm{rad}(G)}{\alpha+2}$$

□

Demaine et al. showed in [3] that the diameter of equilibrium graphs is bounded by $\mathcal{O}(1+\alpha 4^{\sqrt{\lg n}})$ and by $\mathcal{O}(1+(n\alpha^2)^{1/3})$.[2] Combining these results with Corollary 1 yields an improved bound for the price of anarchy:

Lemma 2 ([3]). *The diameter of an equilibrium graph is* $\mathcal{O}(1+(n\alpha^2)^{1/3})$.

Theorem 2. *For* $\alpha = \mathcal{O}(1)$ *the price of anarchy is* $\mathcal{O}(1+(n\alpha^2)^{1/3})$.

Corollary 2. *For* $\alpha = \mathcal{O}(n^{-1/2})$ *the price of anarchy is constant.*

Lemma 3 ([3]). *The diameter of an equilibrium graph is* $\mathcal{O}(1+\alpha \cdot 4^{\sqrt{\lg(n)}})$.

Theorem 3. *The price of anarchy is* $2^{\mathcal{O}(\sqrt{\log(n)})}$.

In the following we show that equilibrium graphs that are trees have cost at most a constant times bigger than the cost of a social optimum. Thus if for given α all equilibrium graphs are trees then PoA is constant. We note that a similar result for SumGame has been shown by Fabrikant et al. in [4]. We show in Section 2.1 that for $\alpha > 129$ all equilibrium graphs are trees which shows that PoA for this range of α is constant.

Theorem 4. *The cost of an equilibrium graph that is a tree is less than 4 times the cost of a social optimum.*

Proof. Observe that the claim is trivial when $n \leq 2$, or when $\alpha < 2/(n-2)$ (as then, by Theorem 1, PoA is at most 2). We therefore assume that $n \geq 3$ and $\alpha \geq 2/(n-2)$. Let $T = (V, E)$ be a tree on $n \geq 3$ vertices that is NE. We first show that $\text{diam}(T) \leq 2\alpha + 3$. Let $v \in V$ be a vertex with $D(v) = \lceil \text{diam}(T)/2 \rceil$ (observe that there exists such a vertex). Consider T rooted at v. Let l be a leaf of T at depth $D(v)$. Consider the strategy of l where l buys, additionally to what it does in the equilibrium strategy profile, an edge to v. The usage cost of l is at most $1 + D(v)$ using the new strategy. Its usage cost in the equilibrium strategy is $D(l) = \text{diam}(T)$. As T is NE we can conclude that buying the edge to v is not beneficial and therefore $\alpha \geq D(l) - (D(v)+1) \geq \lfloor \text{diam}(T)/2 \rfloor - 1$. Hence, $\text{diam}(T) \leq 2\alpha + 3$. We now compare the cost of T with the cost of a social optimum G_{OPT}. As $\alpha \geq 2/(n-2)$, a star is a social optimum (Proposition 1). Hence, as $C(G_{\text{OPT}}) = (\alpha+2)(n-1)+1$,

$$\frac{C(T)}{C(G_{\text{OPT}})} \leq \frac{\alpha(n-1) + \text{diam}(T)\cdot n}{(\alpha+2)(n-1)} \leq \frac{\alpha}{\alpha+2} + \frac{(2\alpha+3)\cdot n}{(\alpha+2)(n-1)} < 1 + 2\cdot\frac{3}{2} = 4,$$

which proves the claim. □

[2] In fact, [3] claims a bound of $\mathcal{O}(\alpha 4^{\sqrt{\lg n}})$ resp. $\mathcal{O}((n\alpha^2)^{1/3})$ on the diameter, which does not make sense for very small α. The arguments given in [3] show a bound of $\mathcal{O}(1+\alpha 4^{\sqrt{\lg n}})$ resp. $\mathcal{O}(1+(n\alpha^2)^{1/3})$ on the diameter.

2.1 For $\alpha > 129$ Every Equilibrium Graph Is a Tree

In this section we present the main result for MaxGame, namely, we show that for $\alpha > 129$ every equilibrium graph is a tree. This, together with Theorem 4, shows that PoA is smaller than 4 for this range of α. The main idea is to show that an arbitrary (non trivial) biconnected component of an equilibrium graph has average degree $c > 2$ and at the same time smaller than $2 + \frac{c'}{\alpha}$ for some constants c, c'. For big enough α these inequalities become contradicting and thus we know that this cannot happen, i.e., every NE for such α contains no biconnected component other than bridges and therefore no cycle – it has to be a tree.

For the entire section let $G = (V, E)$ be a graph on n vertices that contains at least one cycle and let $H \subseteq G$ be an (arbitrary) biconnected component of G of size $|H| \geq 3$. Furthermore we use the following definitions. For a vertex $v \in V$ and a set $X \subseteq V$ we call a path starting in v and ending in a vertex in X a v-X-path. For every vertex v in H we define $S(v)$ to be the set of all vertices $x \in V$ such that a shortest x-H-path ends in v. Note that by definition: $S(v) \neq \emptyset$ since $v \in S(v)$; v is the only vertex from H in $S(v)$; $S(u) \cap S(v) = \emptyset$ for $u \in V(H)$, $u \neq v$; for every $w \in S(v)$ every shortest u-w-path contains v. We start with the observation of Demaine et al. [3] stating that there are no "short" cycles in equilibrium graphs.

Lemma 4 ([3]). *Every equilibrium graph has no cycle of length less than* $\alpha + 2$.

The following lemma shows that the usage cost of vertices in H differ by at most 4 and "tends to be lower" for a vertex that buys an edge in H.

Lemma 5. *If* G *is an equilibrium graph and* $v \in V(H)$ *then* $D_G(v) \leq rad(G)+3$ *if* v *buys an edge in* H *and* $D_G(v) \leq rad(G) + 4$ *otherwise.*

Proof. We show that for every edge $\{u, v\} \in E(H)$ bought by u we have $D_G(u) \leq \text{rad}(G) + 3$ and $D_G(v) \leq \text{rad}(G) + 4$. The claim then follows. Consider a BFS-tree T rooted in some central vertex v_0 of G. First we consider the case that $\{u, v\} \in E(H) \setminus E(T)$. Trivially, $D_G(u) \leq \text{rad}(G) + 1$ (as otherwise u could buy an edge $\{u, v_0\}$ instead of $\{u, v\}$ and thus improve its cost) and therefore $D_G(v) \leq \text{rad}(G) + 2$. Next we consider the case that $\{u, v\} \in E(T) \cap E(H)$. We note that the edge either leads "up" the tree to v_0, or it leads "down" the tree such that there is a vertex $s \in V(H)$ below or at v which is incident to an edge in $E(H) \setminus E(T)$ (if not then u would be a cut vertex of H). In the first case we have $D_G(u) \leq \text{rad}(G) + 1$ (as otherwise u could buy an edge $\{u, v_0\}$ instead of $\{u, v\}$ and thus improve its cost). In the second case we have, as shown before, $D_G(s) \leq \text{rad}(G) + 2$ and therefore $D_G(u) \leq \text{rad}(G) + 3$ (as otherwise u could buy an edge $\{u, s\}$ instead of $\{u, v\}$ and thus improve its cost). So in general we have $D_G(u) \leq \text{rad}(G) + 3$ and therefore $D_G(v) \leq \text{rad}(G) + 4$. □

In the following lemmata we show that for every vertex in a biconnected component H of an equilibrium graph G there is a vertex of degree at least 3 in H in a constant-size neighborhood of v.

Lemma 6. *If G is an equilibrium graph for $\alpha > 0$ then for every vertex v in H and every vertex $w \in S(v)$: $d_G(v,w) \leq rad(G) + \frac{7-\alpha}{2}$.*

Proof. By Lemma 4, H has no cycle of length less than $\alpha + 2$. Thus, as every vertex of H is contained in at least one cycle, there is a vertex $u \in V(H)$ with $d_G(u,v) = d_H(u,v) \geq \lfloor \frac{\alpha+2}{2} \rfloor \geq \frac{\alpha+1}{2}$. Every shortest u-w-path contains vertex v (by definition of $S(v)$). Therefore $d_G(u,w) = d_G(u,v) + d_G(v,w) \geq \frac{\alpha+1}{2} + d_G(v,w)$. By Lemma 5 we have $d_G(u,w) \leq D_G(u) \leq \text{rad}(G) + 4$. Hence $d_G(v,w) \leq \text{rad}(G) + \frac{7-\alpha}{2}$. □

Lemma 7. *If G is an equilibrium graph for $\alpha > 11$, then for every vertex v in H that buys at least two edges in H there is a vertex $w \in N_1(v)$ with $deg_H(w) \geq 3$.*

Proof. Let us refer to v by x_2 and let x_1 and x_3 be two vertices to which x_2 buys edges in H. Assume for contradiction that $\deg_H(x_i) = 2$ for $i = 1,2,3$. Denote the x_1's other neighbor in H by x_0 and the x_3's other neighbor in H by x_4. Note that, as $\alpha > 11$, the girth of H is at least 14 (Lemma 4) and therefore $x_i \neq x_j$ for $i \neq j$. Also by Lemma 6 we have $d_G(x_2, w) < \text{rad}(G) - 1 \leq D_G(x_2) - 1$ for $w \in \bigcup_{i=1,2,3} S(x_i)$. Thus, all shortest x_2-$\mathcal{E}(x_2)$-paths contain either x_0 or x_4. Hence, by buying edges to x_0 and x_4 instead of x_1 and x_3, x_2 would decrease its distance to the vertices in $\mathcal{E}(x_2)$, increase its distance to the vertices in $S(x_1)$ and $S(x_3)$ by at most 1 and it would not increase its distance to any other vertex. Therefore (as $d_G(x_2, w) < D_G(x_2) - 1$ for $w \in S(x_1) \cup S(x_3)$), by changing its strategy x_2 could improve. But this contradicts equilibrium and hence we have $\deg_H(x_i) \geq 3$ for some $i \in \{1,2,3\}$. □

Lemma 8. *If G is an equilibrium graph for $\alpha > 13$ then any path $x_0, x_1, \ldots, x_k$ in H with $deg_H(x_i) = 2$ for $0 \leq i \leq k$ such that for $0 \leq i < k$, $\{x_i, x_{i+1}\}$ is bought by x_i, has length at most $k \leq 4$.*

Proof. Consider a maximal path $x_0, x_1, \ldots, x_k$ in H of the form from the statement and assume for contradiction $k \geq 5$. By Lemma 5 we have $|D_G(x_i) - D_G(x_j)| \leq 3$ for $0 \leq i,j \leq k-1$ and therefore, by the pigeonhole principle, there is $0 \leq i_0 \leq 3$ such that $D_G(x_{i_0}) \geq D_G(x_{i_0+1})$. Denote the x_{i_0+2}'s other neighbor in H by x_{i_0+3} (if not already so denoted). For every vertex $w \in S(x_{i_0+j})$, $j = 0,1,2$, we have (using Lemma 6) $d_G(x_{i_0+j}, w) < \text{rad}(G) - 3$, and therefore $\mathcal{E}(x_{i_0}) \cap S(x_{i_0+j}) = \emptyset$ for $j = 0,1,2$.

We consider the strategy where x_{i_0} buys an edge to x_{i_0+3} instead of the edge to x_{i_0+1} and show that x_{i_0} improves in this strategy, which is a contradiction. We split the vertices of $\mathcal{E}(x_{i_0})$ into two parts: set S where for every $z \in S$ no shortest x_{i_0}-z-path contains x_{i_0+1}, and set $\mathcal{E}(x_{i_0}) \backslash S$ where for every vertex $z \in \mathcal{E}(x_{i_0}) \setminus S$ there is a shortest x_{i_0}-z-path that contains x_{i_0+1} (and therefore also x_{i_0+2} and x_{i_0+3}). Observe that in the new strategy x_{i_0} decreases its distance to vertices in $\mathcal{E}(x_{i_0}) \setminus S$ by 2, and increases its distance to vertices in $S(x_{i_0+1})$ by at most 2, and does not increase its distance to any other vertex of V but perhaps to those in S. We show that x_{i_0} actually decreases its distance to every vertex in S by at least one, which shows that x_{i_0} improves in the new strategy (recall that $d_G(x_{i_0}, y) < D_G(x_{i_0}) - 2$ for every $y \in S(x_{i_0+1})$). To show that x_{i_0} improves its distance to every vertex $z \in S$, we first observe that because $D_G(x_{i_0}) \geq D_G(x_{i_0+1})$ no shortest x_{i_0+1}-z-path contains x_{i_0}. Thus, all shortest x_{i_0+1}-z-paths contain x_{i_0+3}. Hence, in the new strategy, x_{i_0} decreases its distance to z, which finishes the proof. □

Lemma 9. *If G is an equilibrium graph for $\alpha > 13$ then for every vertex v in H there is a vertex $w \in N_5(v)$ with* $\deg_H(w) \geq 3$.

Proof. Let $\{u, v\}$ be an arbitrary edge in H and assume without loss of generality that u bought the edge. Let C be a cycle containing $\{u, v\}$ and note that by Lemma 4 it has at least 16 vertices. Denote the vertices after v and u (in that order) in C by $x_0, x_1, x_2, \ldots$. We distinguish two cases. Assume first that there is a vertex $y \in \{u, x_0, x_1, x_2\}$ that buys both its edges in C. Then, by Lemma 7, there is vertex $w \in N_1(y) \subseteq N_4(u) \subseteq N_5(v)$ with $\deg_H(w) \geq 3$. Assume now that there is no vertex $y \in \{u, x_0, x_1, x_2\}$ that buys both its edges in C. But then, as u buys an edge to v, we have a path x_3, x_2, x_1, x_0, u, v of length 5 where one vertex buys the edge to the next one. Thus, by Lemma 8, the vertices of the path cannot have all degree 2 in H, and the lemma follows. □

Corollary 3. *If G is an equilibrium graph for $\alpha > 13$ then* $\deg(H) \geq 2 + \frac{1}{16}$.

Proof. We assign every vertex $v \in H$ to its closest vertex $c \in H$ with $\deg_H(c) \geq 3$ (thus, c is assigned to itself), breaking ties arbitrarily (by Lemma 9 we know that there is a vertex of degree at least 3 in H). Consider the subgraph of H formed by a vertex c of degree at least 3 and by vertices assigned to it. Observe that these subgraphs form a partition of H. We show that the average degree of every such subgraph is at least $2 + \frac{1}{16}$ which proves the claim. The subgraph consists of $\deg_H(c)$ induced paths $\{p_i(c)\}_{i=1}^{\deg_H(c)}$ that all meet in c. Let $\text{length}(p_i(c))$ denote the length of path $p_i(c)$. By Lemma 9 this length is at most 5. The average degree of the subgraph is then $\frac{\deg_H(c)+2\sum_{i=1}^{\deg_H(c)} \text{length}(p_i(c))}{1+\sum_{i=1}^{\deg_H(c)} \text{length}(p_i(c))} = 2 + \frac{\deg_H(c)-2}{1+\sum_{i=1}^{\deg_H(c)} \text{length}(p_i(c))} \geq 2 + \frac{\deg_H(c)-2}{1+5\cdot\deg_H(c)} \geq 2 + \frac{1}{16}$. □

Next we prove the last ingredient for our approach – we show an upper bound for $\deg(H)$ involving α:

Lemma 10. *If G is an equilibrium graph for $\alpha > 1$ then* $\deg(H) \leq 2 + \frac{8}{\alpha-1}$.

Proof. Consider a BFS-tree T of G rooted in a central vertex $v_0 \in V$ and let $\tilde{T} := T \cap H$. Note that $\tilde{T}$ is a spanning tree of H. Then $\deg(H) = \frac{2|E(\tilde{T})|+2|E(H)\setminus E(\tilde{T})|}{|V(\tilde{T})|} \leq 2 + \frac{2|E(H)\setminus E(\tilde{T})|}{|V(\tilde{T})|}$, and hence we have to bound $|E(H) \setminus E(\tilde{T})|$ (the number of edges outside $\tilde{T}$). To do that, we consider vertices of H that buy an edge in $E(H) \setminus E(\tilde{T})$. Let us call such a vertex a *shopping vertex*. First observe that every shopping vertex u buys exactly one edge in $E(H) \setminus E(\tilde{T})$, as otherwise u could opt not to buy these edges and buy one edge to v_0 instead, thus saving at least α on creation cost, and having usage cost at most $D_G(v_0)+1 \leq D_G(u)+1$, which (for $\alpha > 1$) would be an improvement, a contradiction. This immediately shows that there are at most $|V(\tilde{T})|$ edges in $E(H) \setminus E(\tilde{T})$. To get a better bound, we bound the number of shopping vertices. We show that the distance in $\tilde{T}$ between any two shopping vertices is at least $\frac{\alpha-1}{2}$. The upper bound on the number of shopping vertices follows: Assign every node v of H to the closest shopping vertex (closest according to the distance in $\tilde{T}$; breaking ties arbitrarily); Observe that this assignment forms a partition of H (and that every part contains exactly one shopping vertex); As the distance in $\tilde{T}$ between any two shopping vertices is at least $\frac{\alpha-1}{2}$, the size of every part is at least $\frac{\alpha-1}{4}$. Thus, there are at most $\frac{4|V(\tilde{T})|}{\alpha-1}$ shopping vertices and thus at most that many edges in $E(H) \setminus E(\tilde{T})$; The desired bound $\deg(H) \leq 2 + \frac{8}{\alpha-1}$ now easily follows.

We are left to prove that the distance in $\tilde{T}$ between any two shopping vertices is at least $\frac{\alpha-1}{2}$. Assume for contradiction that there are two shopping vertices $u_1 \neq u_2$ for which $d_{\tilde{T}}(u_1, u_2) < \frac{\alpha-1}{2}$. Let $u_1 = x_1, x_2, \ldots, x_k = u_2$ be the shortest u_1-u_2-path in $\tilde{T}$ and let us call it P. Let $\{u_1, v_1\}$ and $\{u_2, v_2\}$ be the edges that u_1 and u_2 buy in $E(H) \setminus E(\tilde{T})$. Observe that v_1 and v_2 are not descendant of any vertex x_i, $i = 1, \ldots, k$, in P; If v_j, $j = 1, 2$, is descendant of x_i, then the v_j-x_i-path in $\tilde{T}$, the x_i-u_j-path in $\tilde{T}$, and the edge $\{u_j, v_j\}$ form a cycle of length at most $2(d_{\tilde{T}}(u_1, u_2)+1) < \alpha + 1$ which contradicts Lemma 4. In particular, v_j is not part of P, and therefore $x_0 = v_1, x_1, \ldots, x_k, x_{k+1} = v_2$ is a path in H. Also by Lemma 4, u_j, $j = 1, 2$, has distance at least $\frac{\alpha-1}{2}$ from v_0, and therefore v_0 is not in P. Now, since x_1 buys $\{x_0, x_1\}$ and x_k buys $\{x_k, x_{k+1}\}$, there has to be $1 \leq i^* \leq k$ such that x_{i^*} buys both $\{x_{i^*-1}, x_{i^*}\}$ and $\{x_{i^*}, x_{i^*+1}\}$. Consider the following modification of x_{i^*}'s strategy: Buy edge $\{x_{i^*}, v_0\}$ instead of edges $\{x_{i^*-1}, x_{i^*}\}$ and $\{x_{i^*}, x_{i^*+1}\}$. In this new strategy, x_{i^*} decreases its creation cost by α. We now show that x_{i^*}'s new usage cost is $D_{\text{new}}(x_{i^*}) < D_G(x_{i^*}) + \alpha$ thus implying that the new strategy improves x_{i^*}'s cost, a contradiction.

First note that $D_{\text{new}}(x_{i^*}) \leq 1 + D_{\text{new}}(v_0)$ (where the subscript "new" always corresponds to the situation in a graph where x_{i^*} is using the modified strategy). To bound $D_{\text{new}}(v_0)$ we note that only the vertices in P and their descendants in T can have increased distance to v_0 by the strategy change. Let y be one of these vertices with possibly increased distance and let $1 \leq j \leq k$ be such that x_j is the closest ancestor of y, i.e., an ancestor with $d_G(x_j, y) = \min_{x \in P} d_G(x, y)$. If $j = i^*$ it is easy to see that $d_{\text{new}}(v_0, y) \leq d_G(v_0, y)$ and therefore for such a vertex y there is no increase in usage cost of v_0. Consider now the case $j \neq i^*$ and assume (without loss of generality, as we shall see) that $j < i^*$. Then $d_{\text{new}}(v_0, y) \leq d_{\text{new}}(v_0, x_0) + d_{\text{new}}(x_0, x_j) + d_{\text{new}}(x_j, y) = d_G(v_0, x_0) + d_G(x_0, x_j) + d_G(x_j, y)$ (since x_0 is not a descendant of a vertex in P and $x_0, \ldots, x_j$ is still a path in G_{new}), and $d_G(v_0, y) = d_G(v_0, x_j) + d_G(x_j, y)$. Then the increase of usage cost of v_0 is: $d_{\text{new}}(v_0, y) - d_G(v_0, y) = d_G(v_0, x_0) + d_G(x_0, x_j) - d_G(v_0, x_j) \leq 2 \cdot d_G(x_0, x_j) \leq 2 \cdot d_G(u_1, u_2) \leq 2 \cdot d_{\tilde{T}}(u_1, u_2) < \alpha - 1$, where the last inequality follows from our assumption $d_{\tilde{T}}(u_1, u_2) < \frac{\alpha-1}{2}$. As y was chosen arbitrary, we have that the increase of usage cost of v_0 is less than $\alpha - 1$ and therefore $D_{\text{new}}(v_0) < D_G(v_0) + \alpha - 1$, which shows $D_{\text{new}}(x_{i^*}) < D_G(x_{i^*}) + \alpha$. □

Using this result we show that only tree equilibria appear for certain α.

Theorem 5. *For $\alpha > 129$ every equilibrium graph is a tree.*

Proof. If G is a non-tree equilibrium for $\alpha > 129$ and H a block in G with $|H| \geq 3$ then we have by Lemma 10 that $\deg(H) \leq 2 + \frac{8}{\alpha-1} < 2 + \frac{1}{16}$, which contradicts Corollary 3 stating that $\deg(H) \geq 2 + \frac{1}{16}$. □

This bound is asymptotically tight. Indeed there is a constant $c > 0$ such that for $\alpha < c$ we have non-tree equilibrium graphs. E.g., for $\alpha \leq 1$, the triangle is an equilibrium graph (we can generalize this to any size $n \geq 3$ of vertices: three stars of size $n/3$, where the three centers of the stars are connected in a triangle, form an equilibrium graph, too). Theorems 5 and 4 thus show the following.

Corollary 4. *For $\alpha > 129$ the price of anarchy is smaller than 4.*

3 Bounding the Price of Anarchy in SumGame

In this section we consider SumGame. Adapting the methods that we have developed for MaxGame in Section 2.1 we are able to show that in SumGame for $\alpha > 273n$ every equilibrium graph is a tree. This improves the best known bound of $\alpha \geq 12n \log n$ from [1] and is asymptotically the best obtainable bound as for $\alpha < n/2$ there exist non-tree equilibrium graphs [1]. As a corollary we obtain constant PoA for $\alpha > 273n$. We omit most of the proofs and refer to [8], the full version of this paper, for missing details. We use the same conventions and notation as in Section 2.1.

Similarly to MaxGame we can show in the following lemmata that in a constant-size neighborhood of every vertex v in a biconnected component H of an equilibrium graph G there is a vertex of degree at least 3 in H. The details of the proofs are for SumGame a bit different though.

Lemma 11 ([1]). *Any equilibrium graph has no cycle of length less than $\frac{\alpha}{n} + 2$.*

Lemma 12. *If G is an equilibrium graph and $u, v \in V(H)$ are two vertices in H with $d(u, v) \geq 3$ such that u buys the edge to its adjacent vertex x in a shortest u-v-path and v buys the edge to its adjacent vertex y in that path then either $\deg_H(x) \geq 3$ or $\deg_H(y) \geq 3$.*

Proof. Assume for contradiction that $\deg_H(x) = 2 = \deg_H(y)$. Assume without loss of generality that $|S(x)| \leq |S(y)|$. Let z be the other vertex in H adjacent to x. Consider a modified strategy of u where u buys an edge to z instead of the edge to x. In this strategy u shortens its distance to the vertices in $S(y)$ and $S(v)$ by at least 1 and increases its distance to the vertices in $S(x)$ by 1. Furthermore it does not increase its distance to any other vertex in the graph. Since $|S(x)| < |S(v) \cup S(y)|$ ($S(v) \neq \emptyset$ by definition), we conclude that u decreases its cost in the modified strategy, a contradiction. □

The proof of the following lemma is relatively technical (and most different from the techniques used for MaxGame) and it has been omitted due to space reasons.

Lemma 13. *If G is an equilibrium graph then any path $x_0, x_1, \ldots, x_k$ in H, where $\deg_H(x_i) = 2$ for $0 \le i \le k$ and x_i buys $\{x_i, x_{i+1}\}$ for $0 \le i \le k-1$, has length at most $k \le 8$.*

Using the previous two lemmas we can show the following.

Lemma 14. *If G is an equilibrium graph for $\alpha > 19n$ then for every vertex v in H there is a vertex $w \in N_{11}(v)$ with $\deg_H(w) \ge 3$.*

Now, quite in the same way as for MaxGame, we can prove the claims that show the main result of the section.

Corollary 5. *If G is an equilibrium graph for $\alpha > 19n$ then $\deg(H) \ge 2 + \frac{1}{34}$.*

Lemma 15. *If G is an equilibrium graph for $\alpha > n$ then $deg(H) \le 2 + \frac{8n}{\alpha - n}$.*

Theorem 6. *For $\alpha > 273n$ every equilibrium graph is a tree.*

Theorem 7 ([4]). *The cost of an equilibrium graph that is a tree is less than 5 times the cost of a social optimum.*

Corollary 6. *For $\alpha > 273n$ the price of anarchy is smaller than 5.*

References

1. Albers, S., Eilts, S., Even-Dar, E., Mansour, Y., Roditty, L.: On nash equilibria for a network creation game. In: Proceedings of the Seventeenth Annual ACM-SIAM Symposium on Discrete Algorithms (SODA), pp. 89–98. ACM, New York (2006)
2. Alon, N., Demaine, E.D., Hajiaghayi, M., Leighton, T.: Basic network creation games. In: Proceedings of the 22nd ACM Symposium on Parallelism in Algorithms and Architectures (SPAA), pp. 106–113. ACM, New York (2010)
3. Demaine, E.D., Hajiaghayi, M., Mahini, H., Zadimoghaddam, M.: The price of anarchy in network creation games. In: Proceedings of the Twenty-Sixth Annual ACM Symposium on Principles of Distributed Computing (PODC), pp. 292–298. ACM, New York (2007)
4. Fabrikant, A., Luthra, A., Maneva, E., Papadimitriou, C.H., Shenker, S.: On a network creation game. In: Proceedings of the Twenty-Second Annual Symposium on Principles of Distributed Computing (PODC), pp. 347–351. ACM, New York (2003)
5. Jackson, M.O.: Social and Economic Networks. Princeton University Press, Princeton (2008)
6. Jackson, M.O., Wolinski, A.: A strategic model of social and economic networks. Journal of Economic Theory 71(1), 44–74 (1996)
7. Lin, H.: On the price of anarchy of a network creation game (December 2003), final class-project
8. Mihalák, M., Schlegel, J.C.: The price of anarchy in network creation games is (mostly) constant. Tech. Rep. 675, Institute of Theoretical Computer Science, ETH Zurich (July 2010),
`http://www.inf.ethz.ch/research/disstechreps/techreports/`