Game Theory - final project

The Price of Anarchy in Network Creation

Games Is (Mostly) Constant

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The Game:
When I think about network creation games I always think about ISP companies and the internet. The game is very simple, there are N companies, we can call them players (later vertices), each player want to be connected to the whole world - all the other players. every player is selfish - there is no way that two (or more) players cooperate. building a connection to another players has a predefined cost, α. the player who build the connection is the one who pay for it. A connection, or as we call it in the rest of the paper, an edge can be created unilaterally, there is no need of an agreement or any help from the target player. A formal definition can be found in the essay.

Every player in the game wants to minimize the following function which expresses the creation cost of the edges and the accessibility to the other players. There are two popular games that are being reserched - SumGame and MaxGame. both games have a cost function for player u:

$$cost_u = \text{creation cost} + \text{usage cost}$$

Where creation cost is the same in both games:

$$\alpha \cdot (\text{number of edges player } u \text{ buys})$$

Usage cost is the one who makes the difference, in sum game the usage cost is:

$$\sum_{v \in V} d(u, v)$$

Which is the sum over all distances from all other players. d(u,v) is the shortest distance between u and v, i.e. the minimum number of edge in the shortest path between u and v.

in MaxGame the usage cost is:

$$\max_{v \in V} d(u, v)$$

Which is the max distance from u to any other player in the graph.

A Nash Equilibrium (NE for short) of the game is a set of players with strategies such that for every player and every strategy which is different from the strategy he chose, the player's cost function is equal or bigger from the original cost function, i.e. no player can lower his cost by changing his strategy when all other players keep their strategies unchanged.

In our game, a POA is very interesting, it has an important implications about the price that society pays in case of selfish players, or selfish ISP companies. We should be convinced that a constant POA indicates about a "good world" where we are not that far from the social optimum. In addition, we shall realize that a NON constant POA can result with a very high price of the society.

previous results:

Fabrikant et al. introduced and defined SumGame. They proved an upper bound O($\sqrt{[?][?]}$) on PoA - they did it by showing that PoA is bounded by the diameter of the equilibrium graph, which is a very interesting and useful distinction. they also showed that every NE which is a tree has constant PoA - this distinction has been used in our (/current) essay, i.e. Mat´uˇs Mihal´ak and Jan Christoph Schlegel's essay.

Albers et al proved that the PoA in SumGame is constant for α = O($\sqrt{n}$) and α $\geq 12nlogn$
Alon et al. were research on a very similar game with the following property: players do not buy edges, but only swap the endpoints of existing edges.

Demaine et al. proved that the PoA is constant for α < $n^{1-[?][?]}$. They introduced and defined MaxGame and in addition prooved some several bounds for the PoA in MaxGame.

The following chart summarizes what was proved until this essay:

MaxGame:

$\alpha =$	0 … $2\sqrt{\log n}$	$2\sqrt{\log n}$ … n	n … ∞
previous	$\mathcal{O}(n^{2/\alpha})$	$\mathcal{O}(\min\{4^{\sqrt{\log n}}, (n/\alpha)^{1/3}\})$	≤ 2

SumGame:

$\alpha =$	0 – 1	1 – 2	2 – $\sqrt[3]{n/2}$	$\sqrt[3]{n/2}$ – $\sqrt{n/2}$	$\sqrt{n/2}$ – $O(n^{1-\epsilon})$	$O(n^{1-\epsilon})$ – $12n\lg n$	$12n\lg n$ – ∞
previous	1	$\leq \frac{4}{3}$	≤ 4	≤ 6	$\Theta(1)$	$2^{O(\sqrt{\log n})}$	≤ 1.5

Mat´u˘s Mihal´ak and Jan Christoph Schlegel results:

For MaxGame they showed that PoA is constant for α > 129 and α = O($n^{[?][?][?]1/2}$), and also prove that PoA is $2^{O(\sqrt{logn})}$ for any α > 0.
In SumGame they proved that for α > 273n all equilibrium graphs has a constant upper bound on PoA for α > 273n. They did it by proving that for for α > 273n all equilibrium graphs are trees, and then used Fabrikant et al. work (see above).

The following charts summarize their results comparing to the previous ones:

MaxGame:

$\alpha =$	0 – $\frac{1}{n-2}$	$\frac{1}{n-2}$ – $\mathcal{O}(n^{-\frac{1}{2}})$	$\mathcal{O}(n^{-\frac{1}{2}})$ – 129	129 – $2\sqrt{\log n}$	$2\sqrt{\log n}$ – n	n – ∞
new	1	$\Theta(1)$	$2^{\mathcal{O}(\sqrt{\log n})}$	≤ 4	≤ 4	≤ 2
previous	$\mathcal{O}(n^{2/\alpha})$	$\mathcal{O}(n^{2/\alpha})$	$\mathcal{O}(n^{2/\alpha})$	$\mathcal{O}(n^{2/\alpha})$	$\mathcal{O}(\min\{4^{\sqrt{\log n}}, (n/\alpha)^{1/3}\})$	≤ 2

SUMGAME:

$\alpha =$	0	1	2	$\sqrt[3]{n/2}$	$\sqrt{n/2}$	$O(n^{1-\epsilon})$	$273n$	$12n \lg n$	∞
new	1	$\leq \frac{4}{3}$	≤ 4	≤ 6	$\Theta(1)$	$2^{O(\sqrt{\log n})}$	< 5	≤ 1.5	
previous	1	$\leq \frac{4}{3}$	≤ 4	≤ 6	$\Theta(1)$	$2^{O(\sqrt{\log n})}$	$2^{O(\sqrt{\log n})}$	≤ 1.5	

An open problem:
As you can see, there is a range of edge-prices for which we do not know a constant upper bound to α = Θ(n). So, technically there are two options. The first is that there is a constant upper bound but we don't know it yet and we didn't proove it yet. The second is that it might be that there is no constant upper bound to α, and there are some games that suffer of a non constant PoA, And that leads us to my project...

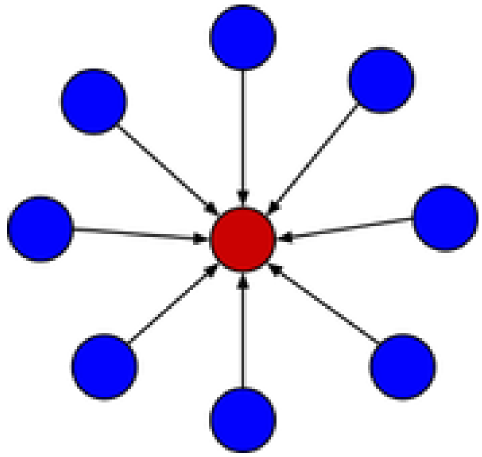
My research:
A great project can be a proof that the price of anarchy is always constant. As we saw, in SumGame we don't know what happens in the range $n \leq$α<273n. Thus, when I thought about an interesting project that can redound to this subject , proving that PoA is constant for every alpha would have been great. However, I wasn't that confident about it - I don't believe that every SumGame has a constant PoA.

Therefore, If we'll think about a stable graph (a NE graph) and α from the edge-prices range that result with a non constant PoA, we'll have a very nice discovery and a well redound to this subject. In my intuitions I believe that there are NE SumGames with a non consant PoA. Therefore, if I can't proof that SumGame's PoA is always constant, I'll try to give a contradict example - a topology of a network that is a NE and under SumGame rules gives us a NON constant PoA.

Then I started to think about topology that seems to be NE and more important - has the property of a non constant PoA. so, I was trying to think about many many topologies with a PoA that seems not constant. When you think about a topology, the difficult thing is to calculate its PoA. calculating the PoA may be hard and not intuitive, so you must do some calculation that may result with a not relevant PoA, and that's what happend to me - I looked for many topologies, and I found some that are NE but when I was trying to calculate their PoA, I realized that the topologies are not good for my purpose. one of the most promising topology is the following one. I'll introduce you a topology, that in the beginning I believed that it can be a contradiction example to the above theorem - a stable graph with a non constant PoA.

<u>The “Stars - Circle” topology:</u>

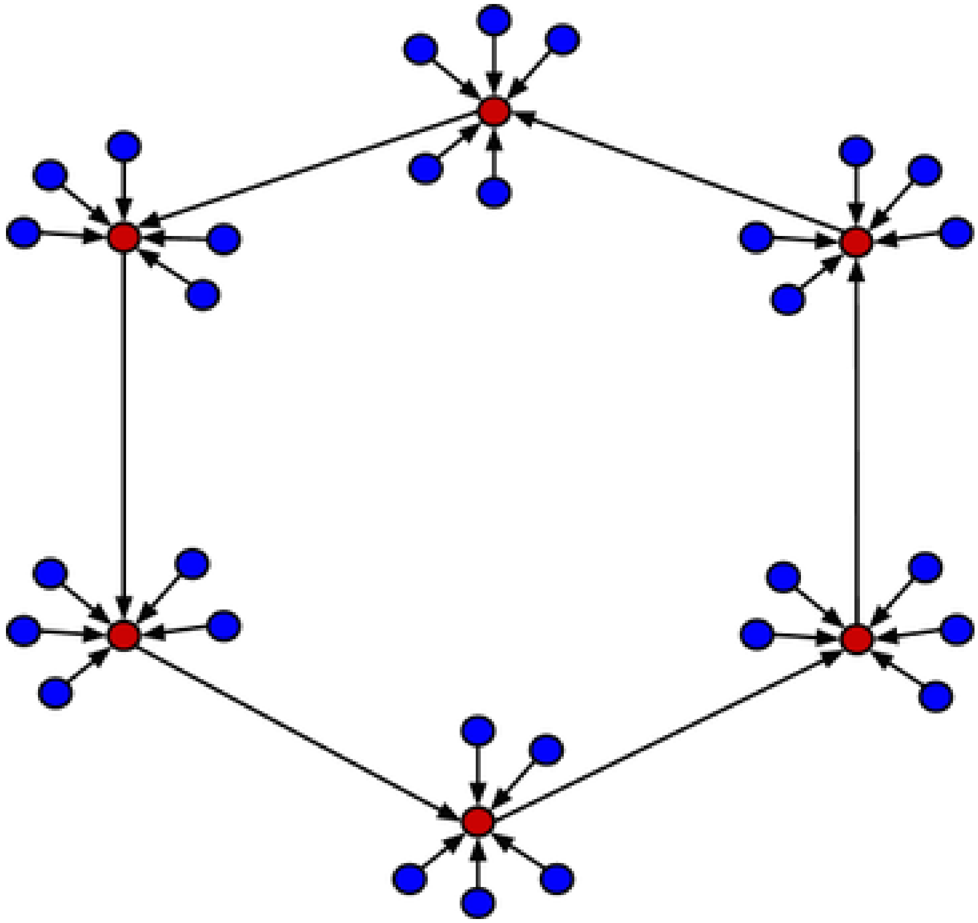
“Stars - Circle” topology is composed of a set of stars that are connected via one edge from their center. Every star in the topology is in the form of:



(1) A single star in “Stars - Circle” topology

Where the red edge is the center of the star and the blue edges are the leafs. Note that the arrow directions define who built the arrow - i.e. in our example, the leafs built the arrows to the star center.

The whole topology is a composition of a group of stars that are connected via the red edges as follows:



(2) The “Stars - Circle” topology structure

Note that each star center has built an arrow to one of his neighbors in order that creates a circle.

A very important property is that given N number of players, every star will be composed of $\sqrt{N}$ players (i.e. $\sqrt{N}$ edges) and the topology will contains $\sqrt{N}$ connected stars. Note that in (2) there are 36 edges, each star contains 6 edges (including the star center) and there are exactly 6 connected stars in the graph.

Thereafter, I was convinced that this topology is a NE and I moved to the more difficult part - calculating the PoA in order to know that our topology is relevant. After calculating the PoA and some extra thoughts and tries to prove that this topology is a NE, I've understood that I was wrong and this topology is not a real NE, yet a very “close” one. Maybe the thinking is good and we can find a better topology that is a real contradiction example, but for now I'll introduce you what I did and the main reasons that led me to my thoughts. After many hours that I spent about finding a topology, I still believe that there might be a good one - maybe I just missed it, But I'm truly aware of the chance that there might be a that the PoA is always constant. Well, lets see the calculation so we'll have the feeling about how close is this topology to be a good example of a NE graph with a non constant PoA.

__PoA calculation:__

we'll sum the usage cost over all the nodes and then the creation cost in order to receive the total cost of the society.

Only for convenience, we'll denote k to be $\sqrt{n}$.

creation cost:

each player establish only one edge - αn

Usage cost:

sum access to stars from all stars:

$$k\,2\sum_{i=1}^{k/2} i = 2k\,(k/2)\,\frac{\frac{k}{2}+1}{2} = 2k\left(k^2/8 + k/4\right) = k^3/4 + k^2/2$$

sum access to leaves from all stars

$$k\,2\sum_{i=1}^{k/2} i+1 = 2k\,(k/2)\,\frac{\frac{k}{2}+1+2}{2} = k^3/4 + k^2/2 + k^2 = k^3/4 + 1.5\,k^2$$

sum access to leaves from all leafs (exclude access to itself as another leaf - so do -2 in the end) :

$$(n-k)\,2\sum_{i=0}^{k/2} (k-1)(i+2) = (n-k)\,2\,(k/2)\,\frac{(\frac{k}{2}+2)(k-1)+2(k-1)}{2} =$$

$$= (n-k)\,2\,(k/4)\left[\left(\frac{k^2}{2} - \frac{k}{2} + 2k - 2\right) + 2k-2\right]$$

$$= (n-k)\,2\,(k/4)\left(\frac{k^2}{2} + 3.5k - 4\right)$$

$$= 2(n-k)\left[(k^3/8) + (7k^2/8) - k\right] > k^5/8$$

$$= n^2k/8$$

sum access to stars from all leafs:

$$(n-k)\,2\sum_{i=0}^{k/2} (i+1) = 2(n-k)\,(k/2)\,\frac{\frac{k}{2}+1+1}{2} = 2(n-k)\left(k^2/8 + k/2\right)$$

Therefore, total sum is smaller than: $n^2k/2$ **(I)**

<u>social optimum - we'll calculate the cost of a star</u>

creation cost (star): (n-1)α

usage cost (star): (n-1) + 2(n-2)(n-1) + (n-1) = 2n-2 + (2n-4)(n-1) = 2n-2 + 2$^{n^2}$- 2n -4n +4 =

star to leafs + (all)leaf to leafs + (all) leaf to star

= 2$^{n^2}$ -4n + 2 = 2($^{n^2}$ -2n +1)= 2$^{(n-1)^2}$ < 2$^{n^2}$ **(II)**

Now, we'll observe the following fraction

C is a social cost function

$$\frac{C([?][?][?]Stars - Circle")}{C(social\,optimum)} \geq \frac{C([?][?][?]Circle - Stars[?][?][?])}{C("Star")}$$

$$according\ to\ ([?][?][?]) \Rightarrow \quad \geq \frac{C([?][?][?]Circle - Stars[?][?][?])}{2n^2}$$

$$according\ to\ ([?][?][?]) \Rightarrow \quad \geq \frac{n^2k/2}{2n^2} = \frac{k}{4} = \frac{1}{4}\sqrt{n}$$

$\Rightarrow$ PoA(**"Circle - Stars") >** $\frac{1}{4}\sqrt{n}$

Note that the above lower bound is smaller than the upper that was proved in previous work. So, in that point I was really convinced that I found something interesting... and then I tried to prove that "Stars - Circle" topology is a real NE.

Trying to proof that its a NE:

We have two types of nodes (or players): a. star node b. leaf node

Consider a player in our "**Stars - Circle**" topology with a strategy s, it's trivial that he can't remove the only edge that he built - otherwise his cost function would have been infinity. Therefore, we will only prove that any player in the **"Stars - Circle**" topology has no better strategy s' that holds s' $\neq$ s.

Now, If we calculate the cost of a player who build an extra edge to the most far node we'll see that its not a better strategy, there are many Summation that result with only one conclusion - the player must build only one edge! **I did these calculations on a paper but I'll spare you from having to read it.**

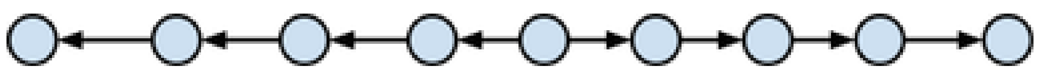
If the best strategy of every player is to build only one edge, we can think about another strategy of the player - building this edge to another star. Therefore, **if a leaf player would have build**

his only edge to another star he will get a lower cost function, because now he has $\sqrt{n}$ close neighbours instead of $\sqrt{n}$-1 close neighbours. In other words, in sumGame, a leaf won't build an edge to his star center - he will build an edge to another star center that has one extra leaf (with him)!

The above "-1" is a small but very important thing that makes **"Stars - Circle**" topology to be irrelevant due to the fact that its simply not a NE graph.

After messing with this topology and understanding that its not a good one I continued to think about other topologies. Among the many that I thought about, there was one basic topology that I want to show to you in order to demonstrate that even with a simple topology, it's not easy to see in advance what is the PoA of the topology. or under which α a graph is NE.

The next topology is a NE given a good α (the proof is very simple). let's call it "Row" topology and demonstrate it's concept using a simple example:



Row topology

Every arrow direction indicates about its owner, for example, the middle vertex built two edges - one edge to its left neighbour and one edge to its right neighbour. Note that the vertices in the corners haven't built any edge.

NE proof:
The vertex in the corners pays the most expansive usage-cost, therefore, we'll analyze their cost and we'll proof that they don't have a better strategy. after that we will conclude that all the other vertices has no better strategy too.

Observe the rightmost vertex. its creation cost is 0, its usage cost is as follows:

$\sum_{i=1}^{n-1} i$ = n($\frac{n}{2}$) = $0.5n^2$

Any other strategy of the rightmost vertex is to build at least one extra edge to another vertex. it is easy to see the among all these strategies, the best one is to build an edge to the middle vertex.lets calculate the cost function of the rightmost vertex in this scenario.
creation-cost: 2α

usage-cost: $2 \cdot \sum_{i=1}^{0.25n} i$ + $\sum_{i=2}^{0.5n} i$ = 0.25n (0.25n+1) + 0.25n (0.5n+1)

= 0.5n + $\frac{1}{16}n^2$ + $\frac{1}{8}n^2$ = 0.5n + $\frac{3}{16}n^2$

Therefore, a Row topology is NOT a good example, because in order to make this topology a NE we need to define α to be O(n^2) and in this case we won't have a non constant POA (because it was already proved that in SumGame for α > 273n all equilibrium graphs are trees. and the POA is constant).

Summarize and few words about the project

Trying to find a contradiction example with a non constant PoA is not that easy - it can be that there is no such an example, and it can be that I just didn't find a good one. It was very hard for me to think of all of this alone - due to some circumstances I didn't share my thought with no one (even friends from the course). I guess that with a little help or some Brainstorming there might be a chance to find a good topology that meets all the requirements. Actually I find the whole subject very interesting, I wish I had more time to Invest in the project - thus I could keep seeking for a good topology, or after convincing myself that there is not such a thing, trying to prove that the PoA is always constant in SumGame (or even shrinks the range of edge-prices for which we do not know a constant upper bound to α = Θ(n)).

bibliography

http://www.math.uwaterloo.ca/~cswamy/courses/co759/material/demainehmz-netcreate.pdf
http://www.cs.tau.ac.il/~fiat/cgt12/Mihalak.pdf - this essay