Emek et al presented a model of probabilistic single-item second price auctions where an auctioneer who is informed about the type of an item for sale, broadcasts a signal about this type to uninformed bidders. They proved that finding the optimal (for the purpose of generating revenue) pure signaling scheme is strongly NP-hard. In contrast, we prove that finding the optimal mixed signaling scheme can be done in polynomial time using linear programming. For the proof, we show that the problem is strongly related to a problem of optimally bundling divisible goods for auctioning. We also prove that a mixed signaling scheme can in some cases generate twice as much revenue as the best pure signaling scheme and we prove a generally applicable lower bound on the revenue generated by the best mixed signaling scheme.

1. INTRODUCTION

Background. Emek et al [2011] recently introduced the following probabilistic single-item auction model: An auctioneer sells a single item to n bidders. The item comes from one of m different types, and the valuations of the bidders for the item vary between the different m types, with the valuation \( v_{ij} \) of bidder i for an item of type j being common knowledge (or at least known to the auctioneer). The actual type of the item is determined by nature, with the probability \( p_j \) of each type j occurring also being common knowledge. There is asymmetry of information in the setting in one respect only: The auctioneer knows the realization of the type of the item, whereas the bidders do not. The auction proceeds by the auctioneer broadcasting to the bidders a single signal about the type of the item. In the work of Emek et al, the signaling schemes considered are pure. That is, the signal is simply some function of the type of the item and in particular, there is a one-to-one correspondence between signaling schemes and partitions of the set of types. After receiving the signal, the bidders bid for the item in a standard 2nd price sealed-bid auction. It is assumed that bidders are risk neutral and play the dominant strategy of bidding their expected valuation given their signal in this auction. Emek et al investigated the following question: To which...
extent can the auctioneer exploit her informational advantage to increase revenue by choosing the signaling scheme appropriately?

Emek et al. show examples where non-trivial schemes significantly outperform the two trivial ones (which are: fully revealing the type of the item and revealing nothing at all). They show that it is strongly NP-hard to compute the pure signaling scheme that maximizes revenue among all such schemes. Their main result is a polynomial time algorithm that finds a pure signaling scheme that approximates the revenue of the optimal one within a constant factor.

Our Results. In this work, we consider the extension of the model of Emek et al. consisting of allowing the auctioneer to use a mixed signaling scheme. In such a scheme, the auctioneer, after witnessing the realization of the item, picks a signal at random according to some probability distribution depending on this realization. We show that by making this very natural extension of the model, we kill two birds with one stone:

— **We earn more:** We show that there are problem instances (with arbitrarily many bidders) where the optimal mixed signaling scheme generates twice the revenue generated by the optimal pure signaling scheme. Also, we show that the revenue generated is never less than $B/2$, where $B = \min_i \left( \sum_j \max_{i' \neq i} p_{ij}v_{i,j} \right)$. We postpone to Section 5 a detailed discussion as to why this particular benchmark is meaningful.

— **We work less:** We show that the optimal mixed signaling scheme can be found in polynomial time, by devising a concise linear program describing this optimal scheme. While it is certainly intuitive that linear programming should be used to find an optimal mixed strategy, we need to prove several structural results concerning the optimal solution before being able to devise a polynomial sized linear program in the present setting.

Discussion of the Model. We are aware that in the setting of Emek et al. (which is our setting as well), having the valuations known to the auctioneer makes it is less than obvious why the model requires the item to be sold in a 2nd price auction. Indeed, simply posting an appropriately chosen price would generate more revenue. Also, the assumption about valuations being known to the auctioneer is itself questionable (note in particular that there is no obvious way to truthfully elicit these valuations from the bidders). To address this critique, we note that Emek et al. use the complete information setup and the associated results outlined above as a component in an analysis of a Bayesian variant of the setup, where the auctioneer is unaware of the actual valuations and has to base her signaling scheme solely on a probabilistic model thereof. Our mixed signaling variant can replace the original pure one also in this Bayesian variant and will increase its revenue and decrease its computational complexity. (We believe it would be interesting to understand how well such a scheme approximates the revenue of the optimal Bayesian auction in the sense of Myerson [1981] in this setting, and suggest this as a possible topic for future work.) Another, more down-to-earth answer to the critique is that a 2nd price auction is simply a very natural, well-known and wide-spread scheme for selling an item and that it therefore makes sense to fix this part of the setup when the main agenda is to investigate how signaling can improve revenue. In essence, our setting allows us to give an exact quantification of the gain the auctioneer can obtain by optimally leveraging her informational advantage. And, as discussed in Section 5, if the valuations are not dominated by a single bidder, then our benchmark-approximation analysis shows that the revenue from 2nd price auctions is comparable with the revenue of the posted price scheme.

Related Research. Due to our setup being a variant of the setup of Emek et al., we refer to their paper for an extensive discussion regarding works dealing with sellers
exploiting their informational advantage (dating back to the Nobel Prize winning work of Akerlof [1970]). However, unlike their pure-signal scheme, our mixed-signal scheme has an alternative interpretation as a model in which the auctioneer sells \( m \) divisible goods to \( n \) bidders which have simple linear valuations per item, by bundling subsets of these goods together (see Section 2.2). The problem of bundling goods, including divisible goods, has received considerable attention in the economics literature (e.g., [Adams and Yellen 1976]) as well as the problem of auctioning divisible goods (see [Back and Zender 2001; Ausubel and Cramton 2004; Iyengar and Kumar 2008] and the books [Cramton et al. 2010; Klepperer 2004]). However, our particular model does not seem to have been considered.

**Organization of Paper.** First, in Section 2, we provide the details of our mixed-signals model and demonstrate that it is equivalent to an auction model concerning bundling of divisible goods. In Section 3, we present the examples where sending mixed signals significantly increases revenue. Then, in Section 4 we show that it is feasible to devise the optimal mixed-signals scheme in poly-time, using a polynomial size LP. Finally, we show in Section 5 that the revenue of the mixed signal auction is at least half the benchmark \( B \). We conclude with discussion and open problems in Section 6.

2. PRELIMINARIES – THE MODEL

2.1. The Problem Formalization

In a probabilistic single-item auction with mixed signals, an auctioneer wants to sell an item drawn from a known distribution \( \{p_1, p_2, \ldots, p_m\} \) over \( m \) types. There are \( n \) bidders that wish to purchase the item, each with valuation \( v_{i,j} \) for an item of type \( j \), with these valuations being common knowledge. The auctioneer observes the type of the item and broadcasts a signal to the bidders. The signaling scheme is strategically chosen by the auctioneer in advance and is given by a map \( \varphi : [m] \times \mathbb{N} \rightarrow [0,1] \), such that for every \( j \), the auctioneer declares signal \( S \) with probability \( \varphi(j,S) \). As we later show, the overall number of signals we send can be assumed to be finite in the scheme generating the largest revenue, so we can assume that from some signal \( B \) and onwards, the function \( \varphi \) is identically 0 (formally, \( \forall j \) and \( \forall S \geq B \), \( \varphi(j,S) = 0 \)). We abuse notation and identify \( S \) with its support (i.e., the set \( \{ j : \varphi(j,S) > 0 \} \)). We also alternate between the notations \( \varphi(j,S) \) and \( \varphi_j,S \). We denote \( S_\varphi \) as the set of all possible signals, i.e. \( S_\varphi = \{ S : \varphi(j,S) > 0 \text{ for some } j \} \). After receiving the signal, the bidders participate in a 2nd price auction for the item.

A pure signaling scheme is one where \( \varphi(j,S) \in \{0,1\} \) for all \( j,S \). The variant of the above setup where the auctioneer is restricted to use a pure signaling \( \varphi \) is the probabilistic single-item auction with pure signals originally suggested by Emek et al. Let us repeat a derivation from Emek et al. for the more general mixed case. For a fixed signal \( S \), the probability of the auctioneer broadcasting this signal is \( \sum_j p_j \varphi(j,S) \), and so, given that the auctioneer broadcasted the signal \( S \), the probability that the item is of type \( j \) is \( \Pr[j|S] = p_j \varphi(j,S)/\left(\sum_{j'} p_{j'} \varphi(j',S)\right) \). As a result, given signal \( S \), the adjusted valuation of bidder \( i \) over the item is \( E[v_i|S] = \sum_j \Pr[j|S] v_{i,j} = \sum_j v_{i,j} p_j \varphi(j,S)/\left(\sum_{j'} p_{j'} \varphi(j',S)\right) \). We assume risk neutral bidders, who follow the dominant strategy of bidding this adjusted valuation in the 2nd price auction. Therefore, for signal \( S \), the auctioneer’s revenue is

\[
\max_i \{E[v_i|S]\} = \max_i \left\{ \frac{\sum_j v_{i,j} p_j \varphi(j,S)}{\sum_j p_j \varphi(j,S)} \right\}.
\]
where $\max_2$ denotes the second highest value. We are interested in the $\varphi$ that maximizes the expected revenue:

$$\maximize \sum_{S \in S_2} \Pr[S] \max_2 \{E[v_i|S]\}$$

$$= \sum_{S \in S_2} \max_2 \left\{ \sum_j (v_{i,j}p_j) \varphi(j, S) \right\} = \sum_{S \in S_2} \max_2 \left\{ \sum_j \psi_{i,j} \varphi(j, S) \right\}$$

where the last equality merely comes from introducing the definition $\psi_{i,j} = v_{i,j}p_j$.

### 2.2. Equivalent Model of Divisible Goods

We observe that a probabilistic single-item auction with mixed signals can alternatively be seen as an auction where $m$ divisible goods are bundled and sold. The mixed signals are crucial for this characterization. The alternative model may be defined as follows: The auctioneer wishes to sell $m$ heterogeneous divisible goods to $n$ bidders. She has 1 unit of each of the goods (for example, she has 1 kilogram from each of $m$ exotic spices). Each bidder $i$ has linear valuation of $\psi_{i,j}$ for each unit of good $j$, so bidder $i$ has a utility of $\sum_j x_j \psi_{i,j}$ if he receives $x_j$ units of each good $j$. The auctioneer sells her goods by bundling several goods together. More precisely, she uses a bundling scheme $(S, \varphi)$, where in each bundle $S \in S$, she places $\varphi_{j,S}$ units of good $j$, and then she runs a 2nd price auction for each bundle. We assume that bidders follow their dominant strategy of bidding their valuation for the bundle for sale in each of these auctions.

The analogy between signaling in the model of one good of $m$ different types, and bundling in the model of $m$ divisible goods, is clear. Given a probabilistic single-item auction with $n$ bidders and $m$ types, we can define a divisible goods auction with $n$ bidders and $m$ goods by letting $\psi_{i,j} = p_jv_{i,j}$. Conversely, given a divisible goods auction with $n$ bidders and $m$ goods, we can define a probabilistic single-item auction with $n$ bidders and $m$ types by letting $(p_j)_{j=1}^m$ be an arbitrary probability distribution with $p_j > 0$ for each $j$ and letting $v_{i,j} = \psi_{i,j}/p_j$. Also, mixed signaling schemes in the probabilistic single-item auction and bundling schemes in the divisible goods auctions are syntactically the same objects. Finally, it is readily checked that the expected revenue in the probabilistic single-item auction is identical (up to a scaling factor) to the revenue in the corresponding divisible good auction. Therefore, finding an optimal mixed signaling scheme in the first model is equivalent to finding an optimal bundling scheme in the latter.

As a result of the above, we allow ourselves the liberty to alternate between the two models.

### 3. Earning More by Sending Mixed Signals

A simple example where the best mixed signaling scheme outperforms the best pure signaling scheme is the following. Assume it is the case where $m = n = 3$, the item is equally likely to be any one of the three types, and the valuations are the identity matrix (bidder $i$ wants only item of type $i$, so $v_{i,i} = 1$, and no other type, so $v_{i,j} = 0$ when $i \neq j$). A pure signaling scheme is forced to pair two of the three types, and results in expected revenue of $\frac{1}{6}$. In contrast, a mixed signaling scheme may use all 3 signals $\{1,2\}, \{1,3\}, \{2,3\}$, and declare any signal that type $j$ belongs to with equal probability. (E.g., if the item type is 1, then with probability $\frac{1}{2}$ the auctioneer declares $\{1,2\}$ and with probability $\frac{1}{2}$ she declares $\{1,3\}$.) Now, no matter what cluster $\{j,j'\}$ was declared, both bidder $j$ and bidder $j'$ know there’s a 50% chance that the item is of their desired type, resulting in a bid of $\frac{1}{2}$ from both bidder $j$ and bidder $j'$. Thus,
the auctioneer gains revenue of $\frac{1}{2}$ with a mixed signaling scheme, exhibiting a gap of 1.5 between the best mixed signaling scheme and the best pure signaling scheme. By a slightly more complex construction, we can get a gap of 2:

**Theorem 3.1.** For any even number $k$, there is a probabilistic single-item auction with $n = k + 1$ bidders and $m = k + 1$ types so that the optimal mixed signaling scheme has an expected revenue which is twice as big as that of the optimal pure signaling scheme.

**Proof.** Consider the auction with valuations as given in Table I and with nature choosing the type uniformly at random.

A pure signaling scheme can only pair $k$ types, so in such a scheme, with probability $\frac{k}{k+1}$, the auctioneer gains revenue of $\frac{1}{2}$ in the 2nd price auction for each signal. In contrast, consider the mixed signaling scheme with signals $\{0, j\}$, where for every $j$,

\[
\Pr[\{0, j\} \mid \text{type } 0] = \frac{1}{k} \quad \text{and} \quad \Pr[\{0, j\} \mid \text{type } j] = 1.
\]

Now, for every signal, the auctioneer gains revenue of $\frac{k}{k+1}$. \(\square\)

4. **Working Less by Sending Mixed Signals**

We now turn to showing that the mixed signaling scheme generating the largest revenue is polynomial-time computable. To that end, we construct a linear program, whose solution is the optimal signaling scheme. In order to devise the LP, we provide several observations, leading the way to formalization of the LP. But before proceeding to the LP and these observations, we introduce some notation.

4.1. **Notation.**

Given a signal $S$, we denote $w_1(S)$ as the bidder which is the winner of $S$ (the bidder with the highest bid), and $w_2(S)$ as the 2nd highest bidder. Formally (recall that we identify $S$ and its support)

\[
w_1(S) = \arg \max_i \left\{ \sum_{j \in S} \psi_{i,j} \varphi_{j,S} \right\}
\]

\[
w_2(S) = \arg \max_{i \not\in w_1(S)} \left\{ \sum_{j \in S} \psi_{i,j} \varphi_{j,S} \right\} = \arg \max_{i} \left\{ \sum_{j \in S} \psi_{i,j} \varphi_{j,S} \right\}
\]

We call a signal $S$ a singleton if $|S| = 1$. Whenever $S$ is a singleton so $S = \{j\}$ for some $j$, we abbreviate $w_1(j), w_2(j)$. Given $i$, we denote by $d(i)$ the set of types for which the bid of $i$ is the biggest: $d(i) = \{ j : w_1(j) = i \}$. Given a signal $S$, we denote $\text{rev}(S)$ as the revenue the auctioneer gets from a 2nd price auction over $S$. I.e.,

\[
\text{rev}(S) = \sum_{j \in S} \varphi_{j,S} \psi_{w_2(S),j},
\]

the bid of bidder $w_2(S)$. As always, we abbreviate singletons to $\text{rev}(j)$ instead of $\text{rev}(\{j\})$. The overall revenue of the auctioneer from $\varphi$ is

---

**Table I. Valuations of an auction in which the optimal mixed signaling scheme has twice the revenue of the optimal pure signaling scheme.**

<table>
<thead>
<tr>
<th>Bidder</th>
<th>$0$</th>
<th>$1$</th>
<th>$2$</th>
<th>$\ldots$</th>
<th>$k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bidder $0$</td>
<td>$m$</td>
<td>$0$</td>
<td>$0$</td>
<td>$\ldots$</td>
<td>$0$</td>
</tr>
<tr>
<td>Bidder $1$</td>
<td>$0$</td>
<td>$1$</td>
<td>$0$</td>
<td>$\ldots$</td>
<td>$0$</td>
</tr>
<tr>
<td>Bidder $2$</td>
<td>$0$</td>
<td>$0$</td>
<td>$1$</td>
<td>$\ldots$</td>
<td>$0$</td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\ddots$</td>
<td>$\vdots$</td>
</tr>
<tr>
<td>Bidder $k$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$\ldots$</td>
<td>$1$</td>
</tr>
</tbody>
</table>
defined as \( \text{rev}(\varphi) = \sum_{S \in \varphi} \text{rev}(S) \). We break ties arbitrarily, but in a consistent manner.

### 4.2. Naïve LP.

Our first observation is that the problem of finding the optimal signaling scheme can be formalized into an LP, with potentially many variables. Assume \( \varphi^* \) is an optimal signaling scheme. We claim \( \varphi^* \) has only a finite number of signals. The key observation is that the auctioneer has no need for two signals \( S \) and \( T \) s.t. \( \text{supp}(S) = \text{supp}(T) \) and in both \( S \) and \( T \) bidder \( i_1 \) is the winning bidder \( (w_1(S) = w_1(T) = i_1) \) and bidder \( i_2 \) is the 2nd highest bidder \( (w_2(S) = w_2(T) = i_2) \). We prove this observation rigorously.

**Claim 4.1.** Let \( \varphi \) be a signaling scheme, and assume that there exist \( S \) and \( T \neq S \) s.t. \( \text{supp}(S) = \text{supp}(T) \), and both \( w_1(S) = w_1(T) \) and \( w_2(S) = w_2(T) \). We define a new signaling scheme \( \varphi' \) by “merging” \( S \) and \( T \) into a single signal \( S' \), and keeping all other signals unchanged. Formally, let \( \varphi' \) be the signaling scheme s.t.

\[
\forall j, \quad \varphi'(j, S') = \varphi(j, S) + \varphi(j, T), \quad \varphi'(j, S) = \varphi'(j, T) = 0
\]

Then for every bidder \( i \) and item type \( j \), the probability \( i \) gets the item of type \( j \) is identical in \( \varphi \) and \( \varphi' \), and \( \text{rev}(\varphi) = \text{rev}(\varphi') \).

**Proof.** The probability of \( i \) winning item of type \( j \) is exactly the probability that the auctioneer sees that the item is of type \( j \) and then declares a signal \( S \), for which \( i \) has the winning bid. This clearly holds for all bidders but \( w_1(S) = w_1(T) \), as all signals for which the winner isn’t \( w_1(S) \) are declared with the same probability in \( \varphi \) and in \( \varphi' \). The claim then follows from showing that \( w_1(S) \) also has the winning bid for \( S' \).

First, observe that under the signal \( S' \), the bidders bid \( E[v_i \mid S'] = \sum_j v_{i,j} \Pr[j \mid S'] = \sum_j p_{i,j} v_{i,j} \varphi'(j, S') \). Therefore, the order of the bids is determined by the numerator in the last term, as the denominator is the same for all bidders. By definition, for every \( i \) we have that \( \sum_j p_{i,j} \varphi'(j, S') = \sum_j p_{i,j} (\varphi(j, S) + \varphi(j, T)) \), and so \( w_1(S) \) had the winning bid for \( S' \) and \( w_2(S) \) has the second highest bid in \( S' \). This allows us to deduce the first part of the claim.

As for revenue, it is evident that \( \text{rev}(\varphi') - \text{rev}(\varphi) = \text{rev}(S') - (\text{rev}(S) + \text{rev}(T)) \), and it is also simple to see that

\[
\text{rev}(S') = \sum_{j \in S'} \varphi'(j, S') \psi_{w_2(S), j} = \sum_{j \in S} \varphi(j, S) \psi_{w_2(S), j} + \sum_{j \in T} \varphi(j, T) \psi_{w_2(T), j} = \text{rev}(S) + \text{rev}(T)
\]

because \( S \) and \( T \) have the same support as \( T \), and because \( w_2(S') = w_2(S) = w_2(T) \). We deduce \( \text{rev}(\varphi') - \text{rev}(\varphi) = 0 \).

Following Claim 4.1, it is evident that the number of signals in an optimal signaling scheme can be upper bounded by all possible subsets of types and pairs of bidders, so \( |S| \leq 2^m n^2 \). Furthermore, constraining \( i_1 \) to be the winning bidder and \( i_2 \) to be the second highest bidder for signal \( S \), is simply a linear constraints. Therefore, by having a variable per signal and a pair of winning bidders, we get that the optimal signaling scheme is the solution for the following (exponential) LP:

\[
\max \sum_{S \subset [m]} \sum_{i_1 \neq i_2} \sum_{j \in S} x_{j}(S, i_1, i_2) \psi_{i_2, j}
\]
under constraints:
\[
\forall S, \forall i_1 \neq i_2, \forall i \neq i_1, i_2 \quad \sum_j x_j(S, i_1, i_2) \psi_{i_1, j} \geq \sum_{j \in S} x_j(S, i_1, i_2) \psi_{i_1, j} \\
\sum_j x_j(S, i_1, i_2) \psi_{i_2, j} \geq \sum_{j \in S} x_j(S, i_1, i_2) \psi_{i_1, j} \quad (2)
\]
\[
\forall S, \forall i_1 \neq i_2, \forall j \quad \sum_{S : j \in S} x_j(S, i_1, i_2) \psi_{i_1, j} \geq \sum_{j \in S} x_j(S, i_1, i_2) \psi_{i_2, j} \quad (3)
\]
\[
\forall S, \forall i_1 \neq i_2, \forall j \quad \sum_{S : i_1 \neq i_2} x_j(S, i_1, i_2) \leq 1
\]
\[
\forall S, \forall i_1 \neq i_2, \forall j \quad x_j(S, i_1, i_2) \geq 0
\]

Where the constraints in (2) assure \( i_1 \) and \( i_2 \) are the two highest bidders for \( S \), and the constraint in (3) assures \( i_1 \) wins for \( S \). The last two constraints assure \( \varphi \) indeed induces a probability for every \( j \).

Therefore, our goal in the remainder of this section is to show that the number of variables in the LP (1) can be reduced to a polynomial number. We comment that the same principals as in the proof of Claim 4.1 will be repeatedly applied in future claims. From now on, we omit the rigorous description of \( \varphi' \), and merely refer to \( \varphi' \) as the result of merging signals into a single signal / splitting a single signal into multiple signals.

### 4.3. Reducing the Number of Variables in the LP.

Our goal is to show that the number of subsets we need to consider in the abovementioned LP can be reduced to a number polynomial in \( n \) and \( m \). To show this, we follow a series of observations. In order to bound the number of signals needed, we'd ideally like to show that every signal can be split. That is, we would like to take any non-singleton signal \( S \) in \( \varphi' \), and have the auctioneer declare a few signals of smaller support rather than declaring \( S \). If such a thing is always possible, then we can recursively split signals until we're left with only singleton signals.

**Definition 4.2.** Given a signaling scheme \( \varphi \), we call a signal \( S \in S_\varphi \) *splittable* if there exists a partition \( S = S_1 \cup S_2 \cup \ldots \cup S_t \) s.t. \( \sum_{k=1}^{t} \text{rev}(S_k) \geq \text{rev}(S) \). We call a signal *singleton-splittable* if the signal is splittable w.r.t. the partition of the signal into \( |S| \) singleton signals, that is \( \sum_{S \in S} \text{rev}(j) \geq \text{rev}(S) \).

Unfortunately, the existence of such a split is not always possible – some signals are non-splittable. Our claims characterize exactly the cases where this split causes the auctioneer to lose revenue.

**Claim 4.3.** Let \( S \) be a signal in the optimal signaling scheme, which is not singleton splittable. That is, \( \text{rev}(S) > \sum_{j \in S} \varphi_{j,S} \text{rev}(j) \). Then both \( w_1(S) \) and \( w_2(S) \) belong to the set of bidders that win the items of \( S \): \( \{ w_1(j) : j \in S \} \).

**Proof.** Assume that \( w_1(S) \) does not belong to the set \( \{ w_1(j) : j \in S \} \). It follows that for every \( j \), the 2nd highest bid cannot be smaller than the bid of the \( w_1(S) \), and so we achieve the contradiction
\[
\sum_{j \in S} \varphi_{j,S} \text{rev}(j) \geq \sum_{j \in S} \varphi_{j,S} \psi_{w_1(S),j} \geq \sum_{j \in S} \varphi_{j,S} \psi_{w_2(S),j} = \text{rev}(S)
\]

Similarly, if \( w_2(S) \) isn’t a winner for some \( j \in S \), then for any \( j \) we have that the bid of the bidder \( w_2(j) \) is no less than the bid of \( w_2(S) \). The inequality follows: \( \sum_{j \in S} \varphi_{j,S} \text{rev}(j) \geq \sum_{j \in S} \varphi_{j,S} \psi_{w_2(S),j} = \text{rev}(S) \). ⊓⊔
The proof of Claim 4.3 gives the following as an immediate corollary.

**Corollary 4.4.** Let $S$ be a signal s.t. the set $\{w_1(j) : j \in S\}$ contains a single bidder. Then $S$ is singleton-splittable.

**Proof.** If $S$ wasn’t singleton-splittable, then the set $\{w_1(j) : j \in S\}$ would contain at least two distinct bidders. □

Corollary 4.4 allows us to deduce that the non-splittable signals must contain at least two distinct bidders in their set of winners. We next show that non-splittable signals must contain at most two distinct bidders in this set.

**Claim 4.5.** There does not exist a non-splittable signal $S$ with $|\{w_1(j) : j \in S\}| \geq 3$.

**Proof.** Assume the existence of a non-splittable signal $S$ with a set of winners, $\{w_1(j)\}_{j \in S}$, containing at least 3 distinct bidders. From Claim 4.3 we know $w_1(S), w_2(S)$ belong to this set, and wlog we denote them simply as bidders $1 = w_1(S)$ and $2 = w_2(S)$. This allows to denote $\text{rev}(S) = \sum_{j \in S} \varphi_{j}(S, j)$, where the winning bid for $S$ is $\sum_{j \in S} \varphi_{j}(S, j)$. We now show $S$ is splittable.

Let us denote the following two disjoint subsets: $S_1 = S \cap d(1)$, $S_2 = S \cap d(2)$, i.e., the set of types in $S$ that bidder 1 (resp., bidder 2) covet the most. Observe that by assumption, some types in $S$ are not in $S_1 \cup S_2$, so we can consider the partition $S_1(l_1, l_2, l_3) = S_1(l_1, l_2) \cup \bigcup_{j \in S_1(l_1, l_2)} \{\varphi_{j}(S, j)\}$. I.e., we partition $S$ into $|S| - |S_1(l_1, l_2)|$ signals: $|S| - |S_1(l_1, l_2)|$ singleton signals, and one signal for all types in $S_1 \cup S_2$.

First, we consider the revenue of the auctioneer from the singleton signals: $\sum_{j \in S_1 \cup S_2} \text{rev}(j)$. On all such types $j$, neither bidder 1 nor bidder 2 have the highest bid, so the 2nd highest bid is at least as high as the bid of bidder 1 and the bid of bidder 2. Therefore, on $S \setminus (S_1 \cup S_2)$, the auctioneer’s revenue is

$$\sum_{j \not\in S_1 \cup S_2} \text{rev}(j) \geq \max \left\{ \sum_{j \in S \setminus (S_1 \cup S_2)} \varphi_{j}(S, j), \sum_{j \in S \setminus (S_1 \cup S_2)} \varphi_{j}(S, j) \psi_{j, S} \right\}$$

Now we consider the revenue of the auctioneer from the signal $S_1 \cup S_2$, where at least one of the bidders $\{1, 2\}$ doesn’t have the winning bid. Therefore, the 2nd highest bid is at least as high as the bid of bidder 1 or the bid of bidder 2. As a result, $\text{rev}(S_1 \cup S_2) \geq \sum_{j \in S_1 \cup S_2} \varphi_{j}(S, j) \psi_{j, S}$ or $\text{rev}(S_1 \cup S_2) \geq \sum_{j \in S_1 \cup S_2} \varphi_{j}(S, j) \psi_{j, S}$. It follows that the above-mentioned partition of $S$ has revenue which is either $\text{rev}(S_1 \cup S_2) + \sum_{j \not\in S \setminus (S_1 \cup S_2)} \text{rev}(j) \geq \sum_{j \in S} \varphi_{j}(S, j) \psi_{j, S}$ or $\text{rev}(S_1 \cup S_2) + \sum_{j \not\in S \setminus (S_1 \cup S_2)} \text{rev}(j) \geq \sum_{j \in S} \varphi_{j}(S, j) \psi_{j, S}$. Observe that $\sum_{j \not\in S \setminus (S_1 \cup S_2)} \text{rev}(j)$ is the winning bid of $S$, so $\text{supp}(S) \cap d(S_1 \cup S_2) \leq \sum_{j \not\in S \setminus (S_1 \cup S_2)} \text{rev}(j)$, and deduce that in any case $\text{rev}(S_1 \cup S_2) + \sum_{j \not\in S \setminus (S_1 \cup S_2)} \text{rev}(j) \geq \text{rev}(S)$. Contradiction. □

Combining Corollary 4.4 and Claim 4.5 we deduce the following.

**Corollary 4.6.** Let $S$ be a non-splittable signal in $\varphi^*$. Then $\{w_1(j) : j \in S\} = \{w_1(S), w_2(S)\}$, otherwise denoted as $\text{supp}(S) \subset d(w_1(S)) \cup d(w_2(S))$.

Using Corollary 4.6 we deduce the existence of an optimal signaling scheme with exactly two types of signals: either singleton signals, or non-splittable signals. Now, using Claim 4.1 we can take any two non-splittable signals $S, T$ such that $w_1(S) = w_1(T)$ and $w_2(S) = w_2(T)$ and merge them. This follows from the fact that we can always think of $S$ and $T$ as two signals over $d(w_1(S)) \cup d(w_2(S))$, with some elements have 0 probability of declaring $S$ (or $T$).

Using 4.1 and 4.6, we deduce that there exists a signaling scheme that has at most $m + n(n - 1)$ different signals: the singleton signals, and the signals composed from
pairing \(d(i)\) and \(d(i')\) for any two bidders \(i, i'\). Observe that \(d(1), d(2), \ldots, d(n)\) partition the \(m\) different types into disjoint sets, so there can only be \(\min\{m, n\}\) such elements in the partition. We therefore deduce that the optimal signaling scheme has at most \(N = m + \min\{m(m-1), n(n-1)\}\) signals. We can therefore reduce our LP to have \(N\) variables; variables \(x_j\), indicating the probability that the auctioneer sees item of type \(j\); and variables \(y_{(i_1, i_2)}\), indicating the probability that the auctioneer sees item of type \(j \in d(i_1) \cup d(i_2)\) and declares a signal in which \(i_1\) has the highest bid, and \(i_2\) has the second highest bid. Formally, we solve:

\[
\max \sum_j x_j \psi\phi_{w_2(j), j} + \sum_{i_1 \neq i_2} \sum_{j \in d(i_1) \cup d(i_2)} y_{(i_1, i_2)} \psi\phi_{i_1, i_2, j} \tag{4}
\]

under constraints:

\[
\forall i_1, \forall i_2 \neq i_1, \sum_{j \in d(i_1) \cup d(i_2)} y_{(i_1, i_2)} \psi\phi_{i_1, j} \geq \sum_{j \in d(i_1) \cup d(i_2)} y_{(i_1, i_2)} \psi\phi_{i_2, j} \]

\[
\forall i_1, \forall i_2 \neq i_1, \text{ and } i \neq i_1, i_2, \sum_{j \in d(i_1) \cup d(i_2)} y_{(i_1, i_2)} \psi\phi_{i_1, j} \geq \sum_{j \in d(i_1) \cup d(i_2)} y_{(i_1, i_2)} \psi\phi_{i_2, j}, \sum_{j \in d(i_1) \cup d(i_2)} y_{(i_1, i_2)} \psi\phi_{i_2, j} \geq \sum_{j \in d(i_1) \cup d(i_2)} y_{(i_1, i_2)} \psi\phi_{i_2, j} \]

\[
\forall j, \sum_{i_1} \sum_{l_2 \neq i_1 \text{ s.t. } j \in d(i_1) \cup d(i_2)} y_{(i_1, i_2)} \psi\phi_{i_2, j} \leq 1 \]

\[
\forall j, \forall i_1, \forall i_2 \neq i_1, x_j \geq 0, \quad y_{(i_1, i_2)} \geq 0
\]

### 4.4. An Additional Observation

Note that for every \(i_1 \neq i_2\) and every \(j \in d(i_1) \cup d(i_2)\), we have two \(y\)-variables in the LP (4), one for \(i_1\) winning and \(i_2\) coming second, and one for \(i_2\) winning and \(i_1\). We now show that it is enough to use just one variable, indicating a signal in which both \(i_1\) and \(i_2\) give the highest bid.

**Observation 4.7.** There exists an optimal signaling scheme, in which for each non-singleton signal \(S\), the first and the second highest bid are identical.

**Proof.** Assume that for a certain signal \(S\), the bid of \(w_1(S)\) is strictly greater than the bid of \(w_2(S)\). Wlog, denote bidder 1 as \(w_1(S)\) and bidder 2 as \(w_2(S)\). We split \(S\) into two disjoint, non-empty sets \(S_1 = S \cap d(1)\) and \(S_2 = S \cap d(2)\). If either \(S_1\) or \(S_2\) are empty, then Corollary 4.4 shows \(S\) can be split into singleton signals. Define

\[
g = \sum_{j \in S_2} \varphi_{j,S} (\psi\phi_{2,j} - \psi\phi_{1,j}) / \sum_{j \in S_1} \varphi_{j,S} (\psi\phi_{1,j} - \psi\phi_{2,j})
\]

(Note, both the numerator and the denominator or positive.) By assumption, we have

\[
\sum_{j \in S} \varphi_{j,S} \psi\phi_{1,j} > \sum_{j \in S} \varphi_{j,S} \psi\phi_{2,j} \quad \Leftrightarrow \quad \sum_{j \in S_1} \varphi_{j,S} \psi\phi_{1,j} + \sum_{j \in S_2} \varphi_{j,S} \psi\phi_{1,j} > \sum_{j \in S_1} \varphi_{j,S} \psi\phi_{2,j} + \sum_{j \in S_2} \varphi_{j,S} \psi\phi_{2,j} \quad \Leftrightarrow \quad \sum_{j \in S_1} \varphi_{j,S} (\psi\phi_{1,j} - \psi\phi_{2,j}) > \sum_{j \in S_2} \varphi_{j,S} (\psi\phi_{2,j} - \psi\phi_{1,j}) \quad \Leftrightarrow \quad g < 1
\]

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Table II. An example.

<table>
<thead>
<tr>
<th></th>
<th>type 1</th>
<th>type 2</th>
<th>type 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>bidder 1</td>
<td>500</td>
<td>500</td>
<td>0</td>
</tr>
<tr>
<td>bidder 2</td>
<td>499</td>
<td>498</td>
<td>1</td>
</tr>
<tr>
<td>bidder 3</td>
<td>7</td>
<td>3</td>
<td>999</td>
</tr>
</tbody>
</table>

The example demonstrating that $B$ and $\tilde{B}$ are different. $\tilde{B}$ requires we ignore bidder 1, as her bids are the highest. $B$ requires we ignore bidder 3, as ignoring bidder 3 results in the biggest decrease in maximal bids.

So now, define $\varphi'$ to be the signaling scheme where for any $j \in S_1$, the probability of giving the signal $S$ decreases: $\varphi'(j, S) = g \cdot \varphi_{j, S}$, and as a result, the probability of giving the singleton signal $\{j\}$ increases: $\varphi'(j, \{j\}) = \varphi_{j, \{j\}} + (1 - g) \cdot \varphi_{j, S}$. In $\varphi'$, the above derivation shows that the bid of bidder 1 and of bidder 2 are identical. Furthermore, by increasing the probability mass on the singleton signals, the auctioneer can only increase her revenue. □

Following Observation 4.7, we deduce that the number of variables in the LP (and the number of signals in our signaling scheme) can be bounded by $N = m + \min\{\binom{n}{2}, \binom{m}{2}\}$. Furthermore, Observation 4.7 justifies the fact that we repeatedly identify a signal with its support.

5. COMPETITIVENESS AGAINST A BENCHMARK

We show a lower bound for the revenue against a benchmark, and first discuss which benchmarks are reasonable. It is quite clear, especially when viewed as selling $m$ divisible goods, that the auctioneer cannot get more than $\sum_j \max_{i \neq i^*} \psi_{i,j}$. As we are restricted to run a 2nd price auction in the end, this quantity is in general unapproachable, since some bidder might have valuations that are so high that they overshadow all other valuations of all other bidders. We thus define our benchmark as the outcome of “taking a bidder out of the picture”. That is, we ignore the bids of $i^*$, and sum the maximum bid for each type separately. Formally,

$$B = \min_{i^*} \left( \sum_j \max_{i \neq i^*} \psi_{i,j} \right) = \min_{i^*} \left( \sum_{j \in d(i^*)} \max_i \psi_{i,j} + \sum_{j \notin d(i^*)} \max_i \psi_{i,j} \right)$$

Before showing our algorithm is competitive with $B$, let us first discuss the motivation for this benchmark. A classical benchmark for comparison in other prior-free settings is the one the results from omitting the bidder with the highest bid (for the same reasoning mentioned above), see Goldberg et al [2006]. Therefore, one might suggest that the right benchmark for the problem is result of ignoring the bid of the one bidder who covets her set of item types the most. Formally, this other benchmark for the problem is: $\tilde{B} = \sum_j (\max_{i \neq i^*} \psi_{i,j})$ where $i^* = \arg\max_i \left( \sum_{j \in d(i)} \psi_{i,j} \right)$. First, observe that $i^*$ and $i_0$, the bidder for which the benchmark $B$ is obtained, are not necessarily the same, as the example in Table II demonstrate. But let us show that are closely related.

CLAIM 5.1.

$$\tilde{B}/2 \leq B \leq \tilde{B}$$

PROOF. The inequality $B \leq \tilde{B}$ follows from the definition of $B$, so we turn to proving $B \geq \frac{1}{2} \tilde{B}$. Denote $i_0$ as the bidder on which the minimum of $B$ is obtained. Obviously, if $i^* = i_0$, we are done, as both benchmarks are the same. So we assume $i^* \neq i_0$, and we
have
\[ B = \sum_j \left( \max_i \psi_{i,j} \right) - \left( \sum_{j \in d(u)} \left( \max_i \psi_{i,j} \right) - \left( \max_2 \psi_{i,j} \right) \right) \geq \sum_{j \in d(v)} \left( \max_i \psi_{i,j} \right) \]
\[ \tilde{B} = \sum_j \left( \max_i \psi_{i,j} \right) - \left( \sum_{j \in d(v)} \left( \max_i \psi_{i,j} \right) - \left( \max_2 \psi_{i,j} \right) \right) \]
and since, by definition of \( i^* \), we have that \( \sum_{j \in d(v)} \psi_{i,j} \geq \sum_{j \in d(u)} \psi_{i,j} \) then it holds that
\[ \tilde{B} - B = \left( \sum_{j \in d(u)} \left( \max_i \psi_{i,j} \right) - \left( \max_2 \psi_{i,j} \right) \right) - \left( \sum_{j \in d(v)} \left( \max_i \psi_{i,j} \right) - \left( \max_2 \psi_{i,j} \right) \right) \]
\[ \leq \sum_{j \in d(v)} \left( \max_2 \psi_{i,j} \right) + \left( \sum_{j \in d(u)} \left( \max_i \psi_{i,j} \right) - \sum_{j \in d(v)} \left( \max_i \psi_{i,j} \right) \right) \]
\[ \leq \sum_{j \in d(v)} \left( \max_2 \psi_{i,j} \right) \leq \sum_{j \in d(v)} \left( \max_i \psi_{i,j} \right) \leq B \]

\[ \blacksquare \]

We comment that the example in Table II also demonstrates that the 2-factor of Claim 5.1 is essentially tight. Now, having established the connection between \( B \) and \( \tilde{B} \), we compare our signaling scheme with the benchmark \( B \).

**Theorem 5.2.** For any set of valuations \( \psi_{i,j} \), the revenue of our signaling scheme is \( \geq B/2 \).

**Proof.** The proof follows from breaking the revenue of the signaling scheme into two terms: the revenue from singleton signals, and the revenue from non-singleton signals. Given a signaling scheme \( \varphi \), we denote
\[ \text{rev}^S(\varphi) = \sum_j \text{rev}(j) = \sum_j \varphi(j; \{j\}) \psi_{\varphi(j);j} \]
\[ \text{rev}^{NS}(\varphi) = \sum_{|S| \geq 2} \text{rev}(S) = \sum_{|S| \geq 2} \varphi(j; S) \sum_{j \in S} \psi_{\varphi(S);j} \]
We now denote \( \varphi^* \) as the optimal signaling scheme we get from solving the LP in (4). Let’s fix \( S \) to be some non-singleton signal in our scheme. So \( S \) corresponds to a pair of bidders, \( i \) and \( i' \), such that \( S \subset d(i) \cup d(i') \) and \( i \) has the highest bid on \( d(i) \), whereas \( i' \) has the highest bid over the items in \( d(i') \). The revenue the auctioneer gets from signal \( S \) is exactly \( \text{rev}(S) = \sum_{j \in d(i) \cup d(i')} y_j(i, i') \psi_{i,j} = \sum_{j \in d(i) \cup d(i')} y_j(i, i') \psi_{i,j} \). Therefore,
\[ 2 \cdot \text{rev}(S) = \sum_{j \in d(i) \cup d(i')} y_j(i, i')(\psi_{i,j} + \psi_{i',j}) \geq \sum_{j \in d(i) \cup d(i')} y_j(i, i')(\max_i \psi_{i,j}) \]
Summing up the revenue of the auctioneer from all non-singleton signals, we have
\[ \text{rev}^{NS}(\varphi^*) \geq \frac{1}{2} \sum_{S: |S| \geq 2} \sum_{j \in S} \left( \max_i \psi_{i,j} \right) y_j(i, i') = \frac{1}{2} \sum_i \left( \max_i \psi_{i,j} \right) \sum_{S: |S| \geq 2} \varphi^*(j, S) \]
We now turn to bound the revenue from singleton signals, that is, the term $\text{rev}^S(\varphi^*)$.

Let us consider the following procedure, that converts one signaling scheme $\varphi$ into a different one $\varphi'$.

1. Let $j = \arg \min \{ \varphi(j, \{j\}) : \varphi(j, \{j\}) > 0 \}$.
2. Fix some $j'$ s.t. $\varphi(j', \{j'\}) > 0$ and s.t. $w_1(j) \neq w_1(j')$.
3. Define $\lambda = \frac{\varphi(j, \{j\}) - \varphi(j', \{j'\})}{\varphi(j', \{j'\})}$ (obviously, $\lambda \leq 1$).
4. Alter $\varphi$ in the following manner. Introduce a new signal $S_{\text{new}} = \{j, j'\}$ and set

$$
\varphi'(j, S_{\text{new}}) = \varphi(j, \{j\}) \quad \varphi'(j, \{j\}) = 0 \\
\varphi'(j', S_{\text{new}}) = \lambda \varphi(j', \{j'\}) \quad \varphi'(j', \{j'\}) = (1 - \lambda) \varphi(j', \{j'\})
$$

Now, the effect of applying this procedure on a signaling scheme is that $\text{rev}^S$ decreases, yet $\text{rev}^\text{NS}$ increases: $\text{rev}^S(\varphi') - \text{rev}^S(\varphi) = -\varphi_{j,j'} \psi_{w_1(j),j} - \lambda \varphi_{j',j'} \psi_{w_2(j'),j'}$, whereas $\text{rev}^\text{NS}(\varphi') - \text{rev}^\text{NS}(\varphi) = \text{rev}(S_{\text{new}})$. But now, because of $\lambda$, the bids of $w_1(j)$ and the bid of $w_1(j')$ are identical for $S_{\text{new}}$, and therefore, just as shown above, $\text{rev}(S_{\text{new}}) = \frac{1}{2} (\varphi(j, \{j\}) + \lambda \varphi(j', \{j'\}) \psi_{w_1(j'),j'})$.

Given a signaling scheme, we denote $J_\varphi = \{ j : \varphi(j, \{j\}) > 0 \}$, and $I_\varphi = \{ w_1(j) : j \in J_\varphi \}$. It is evident that the above procedure is applicable as long as $I$ contains at least two distinct bidders.

Denote the signaling scheme which we end with by $\bar{\varphi}$, and assume $I_\bar{\varphi}$ contains a single bidder, $i_0$ (the case $I_\bar{\varphi} = \emptyset$ is even simpler). Denote $J_{\text{remain}}$ as all the types that appear as singleton in $\bar{\varphi}$ (and obviously in $\varphi^*$), and observe that $J_{\text{remain}} \subset d(i_0)$.

Repeating the derivation from (5), we get that

$$
\text{rev}^\text{NS}(\bar{\varphi}) \geq \frac{1}{2} \sum_j (\max_i \psi_{i,j})(1 - \bar{\varphi}(j, \{j\}))
$$

Since $\text{rev}^S(\varphi) = \sum_{j \in J_{\text{remain}}} \bar{\varphi}(j, \{j\})(\max_i \psi_{i,j})$ we can conclude and deduce that

$$
\text{rev}(\varphi^*) \geq \text{rev}(\bar{\varphi}) = \text{rev}^\text{NS}(\bar{\varphi}) + \text{rev}^S(\bar{\varphi})
$$
We comment that if $\varphi^*$ or $\bar{\varphi}$ contains no singleton signals, then we have $\text{rev}(\varphi^*) \geq \text{rev}(\bar{\varphi}) = \text{rev}^{\text{NS}}(\bar{\varphi}) = \frac{1}{2} \sum_j \max_i \psi_{i,j}$. In words: if the optimal signaling scheme contains no singleton clusters, then the revenue of the auctioneer is at least half the sum of highest bids over all types. We also comment that the example in the introduction, the one where $m = n$ and the valuations form the unit matrix, exhibit a case where the 2-factor in Theorem 5.2 is tight. Lastly, we comment that Ghosh et al [2007] introduced a different algorithm with similar competitiveness ratio against $B$.\footnote{We thank the anonymous referee for bringing the result of [Ghosh et al. 2007] to our attention.}

6. DISCUSSION AND OPEN PROBLEMS

We have shown that in probabilistic single item auctions, mixed signaling schemes outperforms pure ones, both with respect to revenue and with respect to computational complexity. Furthermore, Observation 4.7 gives us an insight as to the characterization of the optimal signaling / bundling scheme. The auctioneer leverages her informational advantage to bundle goods in a way that maximizes competition among bidders – her non-singleton bundles are exactly those where two (or more) bidders are equal in their utility. In that aspect, our model allows us to precisely quantify the extent for which the seller can shape the demand in order to increase her revenue (rather than the usual concern of truthfully sampling the demand, in the non-full information setting). Needless to say, the notion that an increase in the demand leads to an increase in revenue is a basic principle of microeconomics (e.g. [Mas-Colell et al. 1995]).

Similarly, Observation 4.7 also demonstrates the connection between our signaling scheme and the fractional knapsack problem (see [Kellerer et al. 2004]). In fact, one may view the problem as a version of the knapsack problem – for every pair of bidders $(i, i')$ there are numerous ways of bundling the goods s.t. the bids of $i$ and $i'$ are the same. The auctioneer is therefore faced with the problem of picking a subset of these potential bundle (subject to having at most one unit of each good) in order to maximize her profit. And, much like the fact that the fractional knapsack problem is polynomial time solvable, so is the mixed signals problem.

Finally, we suggest some interesting open problems:

— Are there instances where the optimal mixed signaling scheme generates strictly more than twice the revenue of the optimal pure signaling schemes?

— In Bayesian variants of the setup (see [Emek et al. 2011]), how well does the signaling + 2nd price auction approach approximate the optimal auction (in the sense of Myerson [1981])?

— Is it possible to find an optimal (or approximately optimal) signaling scheme when $m$ is exponentially large? Consider the case where each type can be described using $d$ attributes, and the bidders’ valuations for the item are functions of these $d$ attributes. Can one extend the LP of (4) to handle such valuations?

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