Computational Game Theory **Fall Semester, 2012**

Report: ρ-Approximate Equilibria in Unweighted Congestion Games

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1.1 Introduction

Among other solution concepts, the notion of the pure Nash equilibrium plays a central role in Game Theory. Pure Nash equilibria in a game characterize situations with non-cooperative deterministic players in which no player has any incentive to unilaterally deviate from the current situation in order to achieve a higher payoff. Unfortunately, it is well known that there are games that do not have pure Nash equilibria. Furthermore, even in games where the existence of equilibria is guaranteed, their computation can be a computationally hard task. Such negative results significantly question the importance of pure Nash equilibria as solution concepts that characterize the behaviour of rational players.

Following the above I decided to explore the complexity and existence of approximate pure Nash equilibria, which characterize situations where no player can significantly improve her payoff by unilaterally deviating from her current strategy.

My goal was to make a work progress upon this topic, with respect to the following metrics:

+ Choosing a paper and prepare a short presentation
+ Suggest a research topic based upon the paper
+ Make a progress report

1.2 The Paper

The research is based on the work of Ioannis Caragiannis, Angelo Fanelli, Nick Gravin and Alexander Skopalik: "Approximate Pure Nash Equilibria in

Weighted Congestion Games: Existence, Efficient Computation, and Structure" (Submitted on 12 Jul 2011 (v1), last revised 11 Nov 2011)[1]. In the article, the authors present an algorithm that computes O(1)-approximate equilibria in weighted congestion games, using a new class of potential games, which approximate weighted congestion games with polynomial latency function, $\psi - games$. Each weighted congestion game of degree d ≥ 2 has a corresponding $\psi - game$ of degree d defined in such a way that any ρ-approximate equilibrium in the latter is a $d!\rho - approximate$ equilibrium for the former. The article raises an immediate question regarding using similar approach on unweighted congestion games, focusing on the worst-case ratio of the potential value at an almost exact pure Nash equilibrium over the globally optimum potential value.

The article's algorithm achieves a stretch of $\frac{3+\sqrt{5}}{2}$ for linear weighted congestion games.

1.3 Research Topic: ρ-Approximate Equilibria in Unweighted Congestion Games

The case of unweighted congestion games (i.e., when all players have unit weight) has been widely studied in literature. Rosenthal[2] proved that these games admit a potential function with the following remarkable property: the difference of the potential value between two states (i.e., two snapshots of strategies) that differ in the strategy of a single player equals to the difference of the cost experienced by this player in these two states. This immediately implies the existence of a pure Nash equilibrium. Any sequence of improvement moves by the players strictly decreases the value of the potential and a state corresponding to a local minimum of the potential will eventually be reached; this corresponds to a pure Nash equilibrium.

Nevertheless, potential functions provide only inefficient proofs of existence of

[1] Approximate Pure Nash Equilibria in Weighted Congestion Games: Existence, Efficient Computation, and Structure - `http://arxiv.org/abs/1107.2248v2`

[2] R. W. Rosenthal. A class of games possessing pure-strategy Nash equilibria. International Journal of Game Theory, 2:6567, 1973.

pure Nash equilibria. Fabrikant et al.[3] proved that the problem of computing a pure Nash equilibrium in an unweighted congestion game is PLS-complete (informally, as hard as it could be given that there is an associated potential function; [4]). One consequence of PLS-completeness results is that almost all states in some congestion games are such that any sequence of player's improvement moves that originates from these states and reaches pure Nash equilibria is exponentially long. Efficient algorithms are known only for special cases. For example, Fabrikant et al. showed that the Rosenthals potential function can be (globally) minimized efficiently by a flow computation in unweighted congestion games in networks when the strategy sets of the players are symmetric.

The above negative results upon unweighted congestion games enlarge the motivation toward alternative concepts such as approximate Nash equilibria. Before getting into the thick of things and examining the article's algorithm on unweighted congestion games, I found out that the authors had already dealt with it in their previous article; In 2011 at the 52nd annual IEEE Ioannis Caragiannis, Angelo Fanelli, Nick Gravin and Alexander Skopalik published an article: "Efficient Computation of Approximate Pure Nash Equilibria in Congestion Games"[5], in which they present a surprisingly simple polynomial-time algorithm that computes $O(1)$-approximate Nash equilibria in these games. In particular, for congestion games with linear latency functions, the algorithm computes $(2+\epsilon)$-approximate pure Nash equilibria in time polynomial in the number of players, the number of resources and $1/\epsilon$. It also applies to games with polynomial latency functions with constant maximum degree d: there, the approximation guarantee is $d^{o(d)}$. The algorithm essentially identifies a polynomially long sequence of best-response moves that lead to an approximate equilibrium; the existence of such short sequences is interesting in itself. Furthermore, the authors strengthen the results further by proving that, for congestion games that deviate from their mild assumptions, computing -approximate equilibria is PLS-complete for any polynomial-time computable . Hence, it isn't a surprise that applying

[3]A. Fabrikant, C. H. Papadimitriou, and K. Talwar. The complexity of pure nash equilibria. In Proceedings of the 36th Annual ACM Symposium on Theory of Computing (STOC), ACM, pages 604612, 2004

[4]D. S. Johnson, C. H. Papadimitriou, andM. Yannakakis. How easy is local search? Journal of Computer and System Sciences, 37: 79100, 1988.

[5]Efficient Computation of Approximate Pure Nash Equilibria in Congestion Games - http://ieeexplore.ieee.org/stamp/stamp.jsp?tp=&arnumber=6108214

the extended algorithm, for weighted congestion games, on unweighted congestion games results in a $\frac{3+\sqrt{5}}{2}$ ($> 2+\epsilon$) for linear weighted congestion games, and $d^{2d+o(d)}$ ($> d^{o(d)}$).

1.4 Conclusions and Personal Notes

I find the article and the working progress very interesting, though a solution to the research topic has already been given. The concept of approximating non-potential games by potential ones seems to be extraordinary and might have further applications. As for both weighted and unweighted congestion games, the solution depends on the fact that the latency functions have non-negative coefficients - this limitation is a must because computing approximate equilibria is PLS-complete for congestion games with linear latency functions that have negative offsets[6]. Finally, trying to apply similar techniques on constraint satisfaction games could be applicable.

[6]Efficient Computation of Approximate Pure Nash Equilibria in Congestion Games, Theorem 5.1, p.9 - `http://ieeexplore.ieee.org/stamp/stamp.jsp?tp=&arnumber=6108214&tag=1`