Mechanism Design on Discrete Structures

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August 7, 2011

Abstract

We study strategyproof (SP) mechanisms for the location of a facility on a discrete graph. We give a full characterization of SP mechanisms on lines and on sufficiently large cycles. Interestingly, the characterization deviates from the one given by Schummer and Vohra [21] for the continuous case. In particular, it is shown that an SP mechanism on a cycle is close to dictatorial, but all agents can affect the outcome, in contrast to the continuous case. Our characterization is also used to derive a lower bound on the approximation ratio with respect to the social cost that can be achieved by an SP mechanism on certain graphs. Finally, we show how the representation of such graphs as subsets of the binary cube reveals common properties of SP mechanisms and enables one to extend the lower bound to related domains.

1 Introduction

In many natural settings that include social or economic interactions, the involved parties cannot be assumed to cooperate (e.g. to report private information truthfully when asked to). Rather, participating parties act as self-interested agents, each aiming to maximize her own benefit. The purpose of mechanism design

A mechanism is said to be *Strategyproof* (SP) if it is a dominant strategy for every agent to truthfully reveal her private information, independent of the reports of other agents. E.g., a generic SP mechanism is the Vickrey-Groves-Clarke(VCG) mechanism (see [18]) that achieves SP by allowing monetary transfers.

Unfortunately, arbitrary monetary transfers might be impractical, illegal or even meaningless in many real world situations. In such cases, one is restricted to devise mechanisms that do not involve them. Our primary interest is in characterizing the set of SP mechanisms that do involve monetary transfers in certain domains, sometimes requiring other natural properties on top of truthfulness.

We are also interested in the social welfare that can be guaranteed by SP mechanisms. Since optimal solution (i.e., maximal social welfare) is not always achievable without payments, we care about the best approximation to the optimum that can be guaranteed by SP mechanisms. This agenda of approximate mechanism design without money, namely the study of moneyless SP approximation mechanisms for optimization problems, was recently explicitly advocated by Procaccia and Tennenholtz[20].

The *facility location* problem is a very natural setting, in which agents that are located in some metric space report their locations (which is private information), and a facility location is determined based on these reports, with the purpose of being as close as possible to all agents (on average). For example, consider a school bus that is planned to have a single stop in a certain street (the street will be our metric space). Pupils living in this street are requested to report their exact addresses, and the bus stop is located so as to minimize the average distance of the pupils. If truthfulness can be assumed, then this problem is trivial. However, the optimal mechanism fails if agents are acting strategically. For example, a pupil living close to the end of the street may benefit from reporting a more extreme location, thereby biasing the location of the bus station toward her real address.

PreviousWork

1.1 Previous work

While the most studied setting for Mechanism design without money (MDw/oM) is facility location, SP mechanisms without money have been proposed and analyzed in a wide variety of domains such as matching [22, 3, 9], resource allocation [10, 11, 19], machine learning [5, 15], judgment aggregation [8, 17], and even auctions [12].

The deterministic facility location problem on a *continuous graph* was studied by Schummer and Vohra [21]. They characterized deterministic SP mechanisms on a line, and extended the characterization to trees. They further showed that on a circular graph, every SP mechanism (that is also onto) must be dictatorial. That is, the location of the facility always coincides with the location of the dictator.

This result was later leveraged by Alon et al. [2] to derive a lower approximation bound of n (the number of agents) for the SP facility location problem on a continuous cycle. The authors then demonstrated how a constant approximation can be guaranteed with a randomized mechanism.

Meir et al studied characterizations and approximation bounds for the strategyproof classification problem [13, 15]. While this seems quite unrelated to the problem at hand, a variation of their model (which they call the "realizable" setting) is in fact equivalent to facility location on subgraphs of the binary cube.[1] Meir et al. showed that any deterministic and onto SP mechanism (on some specific domain) *must be dictatorial*, and proved a similar result for randomized mechanisms. Nevertheless this strong negative result has to be shown to still hold in the realizable setting, and to the best of our knowledge, no non-trivial lower approximation bound for discrete facility location problem has been derived.

Our model is closely related to the literature on *voting with single-peaked preferences*. Yet, in our model an agent's location implies not only her peak, but her entire preference structure. Strategyproof mechanisms in the general single-peaked model [4] were characterized by Moulin [16]. Note that the general setting allows for richer preference structures, and thus strategyproofness is a more restrictive requirement in such models.

Contribution

1.2 Our contribution

In this paper, we follow a setting studied by Schummer and Vohra [21] and later by Alon et al. [2] for facility location on graphs. We replace the continuous graph in the original model with a discrete graph, where the agents and the facility are restricted to vertices only. We feel that in practical problems, such as choosing some building in a street to host a facility, a continuous model is inappropriate, and discrete mechanisms should be better understood.

We give an exact characterization of SP (and onto) mechanisms on certain families of discrete graphs, focusing on a discrete line and discrete circular graphs (cycles). We limit our attention to deterministic mechanisms. For both families, we give an embedding of the graph as a subset of the binary cube, which interestingly allows us to express sufficient and necessary properties of SP mechanisms using natural notions.

For large cycles, our characterization implies that every onto SP mechanism is nearly-dictatorial. While this result resembles that of Schummer and Vohra for continuous cycles, we show that the size

[1] The original, non-realizable setting of Meir at al. can be interpreted as a generalization of the facility location problem, where agents may be placed in locations that are forbidden for the facility. See more details on this mapping in Appendix E.

of the cycle matters, and that even in large cycles, there exist SP mechanisms in which all the agents have some level of influence. As a corollary, we get the first lower approximation bound on discrete facility location and show it to be n (i.e., the number of agents). This result also entails a similar lower bound for realizable SP classification problems, thereby showing that the negative result of Meir et al. also holds in a particular case of interest.

2 Preliminaries

reliminaries

Consider a graph $G = \langle V, E \rangle$ with a set V of vertices and a set E of edges, where edges have no weights nor direction. The vertices $v \in V$ will be also referred to as the *locations*, and the two terms will be used interchangeably. The distance between two vertices $v, v' \in V$, denoted $d(v, v')$, is the length of the minimum-length path between v and v', where the length of a path is defined as the number of edges in the path.[2] Note that d is a distance metric. We extend the notion of distance to denote the distance between two sets of vertices, where the distance between two sets is defined as the minimum distance between any vertices in the two sets; i.e., given two sets of vertices $A, A' \subseteq V$, $d(A, A') = \min_{v \in A, v' \in A'} d(v, v')$.

In this paper we will be especially interested in two types of graphs, namely lines and cycles.

Line graphs A line graph with $k+1$ vertices is denoted by $L_k = \{0, 1, \ldots, k\}$. We refer to an increase (respectively, decrease) of the index as a movement in the *right* (resp., *left*) direction. Clearly, in line graphs, every two vertices are connected by a single path. For $v' > v$, let the closed interval $[v, v']$ denote the set of vertices $\{v, v+1, \ldots, v'-1, v'\}$, and let the open interval (v, v') denote the set of vertices $\{v+1, \ldots, v'-1\}$.

Cycle graphs A cycle graph with k vertices is denoted by $R_k = \{0, 1, \ldots, k-1\}$.[3] We refer to an increase (respectively, decrease) of the index as a movement in the *clockwise* (resp., *counter-clockwise*) direction. We denote the closed arc between v and v' in the clockwise direction as $[v, v']$ and the open arc in the clockwise direction as (v, v').

Let $N = \{1, \ldots, n\}$ be the set of agents, and $\mathbf{a} = (a_1, \ldots, a_n) \in V^n$ be a *location profile*, where v_j denotes the location of agent j for every $j \in N$. The locations of all the agents excluding agent j is denoted by a_{-j}.

A *facility location mechanism* (or *mechanism* in short) for a graph $G = \langle V, E \rangle$ and a location profile is a function $f : V^n \to V$, specifying the chosen facility location for every location profile. Agents prefer having the facility located as close to them as possible (and are indifferent between locations of the same distance from them). Interestingly, facility location mechanisms on the k-dimensional cube (or, more accurately, on certain subsets of the cube) are closely related to mechanisms in other domains, such as distance-based judgement aggregation and classification. We shall elaborate on this important topic in Sections 5 and 6.

2.1 Properties of mechanisms

sec:prop

We start with several definitions of mechanism properties, which are independent of the graph topology. While we believe that the reader is familiar with these properties, we provide their definitions for completeness.

Definition 1. *A mechanism f is* onto*, if for every $x \in V$, there exists a profile $\mathbf{a}$ s.t. $f(\mathbf{a}) = x$.*

[2] If v, v' are not connected then $d(v, v') = \infty$, however we only consider connected graphs in this paper.

[3] In some places in the paper we start labeling the vertices from 1.

This property is a very basic requirement (usually justified as society sovereignty), and as such we will restrict attention to mechanisms satisfying this condition. We are also interested in the following properties, which are stronger.

Definition 2. *A mechanism f is* unanimous *if for every $x \in V$, $f(x, x, \ldots, x) = x$.*

def:pareto **Definition 3.** *A location $y \in V$ is said to Pareto dominate $x \in V$ if all the agents strictly prefer y over x (i.e. $d(y, a_j) < d(x, a_j)$ for every $j \in N$). A mechanism f is* Pareto *if for all $\mathbf{a} \in V^n$, there is no location $y \in V$ that Pareto dominates $f(\mathbf{a})$.* [4]

It is easy to verify that Pareto entails unanimity, which in turn entails onto.

An agent j is said to be a *dictator* in f if for every location profile $\mathbf{a} \in V^n$, it holds that $f(\mathbf{a}) = a_j$. Consider the following relaxation of the dictatorship notion.

Definition 4. *An agent j is said to be an m-*dictator *in f, if for every $\mathbf{a} \in V^n$, it holds that $d(f(\mathbf{a}), a_j) \leq m$. A mechanism f is m-dictatorial if there exists an agent j that is an m-dictator in f.*

Note that a 0-dictator is essentially a dictator. It is argued that dictatorial mechanisms are "unfair" in the sense that the agent's name plays a major role in the decision of the facility location. Mechanisms that ignore agents' names altogether are said to be anonymous.

Definition 5. *A mechanism f is* anonymous*, if for every profile $\mathbf{a}$ and every permutation of agents $\pi : N \to N$, it holds that $f(a_1, \ldots, a_n) = f(a_{\pi(1)}, \ldots, a_{\pi(n)})$.*

Our main interest is in *strategyproof* mechanisms, defined as follows.

Definition 6. *A mechanism f is* strategyproof *(SP), if no agent can strictly benefit by misreporting her location; that is, for every profile $\mathbf{a} \in V^n$, every agent $j \in N$ and every alternative location $a'_j \in V$, it holds that*

$$d(a_j, f(a'_j, a_{-j})) \geq d(a_j, f(\mathbf{a})).$$

The following lemma relates some of the aforementioned properties.

emma:onto_un **Lemma 1.** *Every mechanism f that is both onto and SP, is unanimous.*

2.2 The social cost

In addition to the characterization of SP mechanisms, we shall be also interested in the performance of the mechanisms, as evaluated with respect to some well-defined objective function. The social cost function considered in this work is the sum of the distances of the agents' locations from the chosen facility location. That is, given a location profile $\mathbf{a}$ and a facility location x, the social cost is given by $SC(x, \mathbf{a}) = \sum_{j \in N} d(a_j, x)$. [5] When evaluating a mechanism's performance, we use the standard worst-case approximation notion. Formally, given a profile $\mathbf{a}$, let opt$(\mathbf{a})$ be an optimal facility location (i.e., opt$(\mathbf{a}) \in \mathrm{argmin}_{x \in V} SC(x, \mathbf{a})$). We say that a mechanism f provides an α-approximation if for every $\mathbf{a} \in V^n$, $SC(f(\mathbf{a}), \mathbf{a}) \leq \alpha \cdot SC(\text{opt}(\mathbf{a}), \mathbf{a})$.

In some cases, there is a tradeoff between strategyproofness and performance. For example, if one is willing to compromise on truthfulness, the mechanism that returns opt$(\mathbf{a})$ for every $\mathbf{a}$ achieves optimal performance. On the other hand, if one cares only about strategyproofness, the dictatorial

[4] Note that this requirement is slightly weaker than the more common definition of Pareto, requiring that no other location can strictly benefit one of the agents without hurting any other agent.

[5] Other objective functions have been considered in the literature, such as the *egalitarian cost*, aiming at minimizing the maximum cost, see e.g., Alon et al. [2].

mechanism is SP, but it may exhibit a bad approximation of $n-1$ if all the agents but the dictator share the same location. Thus, a natural challenge is to identify cases where SP mechanisms can guarantee good performance. In this paper we restrict attention to deterministic mechanisms. In addition, we assume that possible agent locations and facility locations coincide. In the more general case, the set of allowed facility locations (i.e., the range of f) may be more restricted than the set of possible agent locations V. For example, a bus stop may need to be located on a main road, while the agents can be located anywhere in the city. Clearly, the necessary conditions for strategyproofness provided in this paper, as well as lower bounds on the approximation ratio, carry over to the general case.

3 SP mechanism over a discrete line

sec:line

In this section we provide a characterization of all onto SP mechanisms on a discrete line.

Given an alternative $x \in L_k$, agent j's cost, denoted by $c_j(x)$, is defined to be the distance between x and a_j; i.e., $c_j(x) = d(a_j, x) = |a_j - x|$. The notion of monotonicity follows.

Definition 7. *A mechanism f on a line is* monotone *(MON) if for every $b_j > a_j$, $f(a_{-j}, b_j) \geq f(\mathbf{a})$.*

In other words, monotonicity of a mechanism means that if an agent moves in a certain direction, the outcome cannot move in the other direction as an effect. The following two properties bound the effect of an agent's deviation on the outcome of the mechanism.

Definition 8. *A mechanism f is m-*step independent *(m-SI) if for every $a'_j > a_j$ (respectively, $a'_j \leq a_j$), if $d([a_j, a'_j], f(a)) > m$ (resp., $d([a'_j, a_j], f(a)) > m$), then $f(a'_j, a_{-j}) = f(\mathbf{a})$.*

Definition 9. *A mechanism f is* disjoint independent *(DI) if for every a'_j, if $f(\mathbf{a}) \neq f(a'_j, a_{-j})$, then $|[a_j, a'_j] \cap [x, x']| \geq 2$, where $x = f(\mathbf{a})$ and $x' = f(a'_j, a_{-j})$.*

Intuitively, m-SI means that a deviation that occur in an interval sufficiently far from the original outcome does not affect it. The DI property means that an agent can affect the outcome of the mechanism only if its trajectory intersects the trajectory of the facility in at least two consecutive points.

A mechanism is said to be *strongly m-step independent* (m-SSI) if it is both m-SI and DI. For example, the median mechanism (and in fact any order statistics mechanism) is strongly 0-SSI.

Our first primary result characterizes all the mechanisms that satisfy the requirements of onto and SP on the line.

th:line **Theorem 2.** *An onto mechanism f on the line is SP if and only if it is MON and 1-SSI.*

The remainder of this section is dedicated to sketching the proof of Theorem 2. Full proofs of all lemmas are in Appendix B. An alternative characterization is given in Section 5, using the notations of the binary cube.

lemma:SP_mon **Lemma 3.** *Every SP mechanism is monotone.*

a:mon_pareto **Lemma 4.** *A monotone mechanism f is Pareto iff it is unanimous.*

Notice that the Pareto property (Def. 3) has a simpler form in this domain: $f(\mathbf{a}) \in [\min_{j \in N} a_j, \max_{j \in N} a_j]$.

The following lemma is the main building block in the proof of Theorem 2.

ma:SP_to_IIA **Lemma 5.** *Every SP, unanimous mechanism for the line is 1-SI.*

A few remarks are in order. Lemma 18 in the appendix establishes that every 0-SI monotone mechanism on the line is SP. The following example shows that 0-SI is *not* a necessary condition: Consider a setting with two players and the following mechanism f on L_2: $f(a_1, a_2) = 2$ if $a_1 = 2$ or $a_2 = 2$, $f(a_1, a_2) = 1$ if $a_1 = a_2 = 1$, and $f(a_1, a_2) = 0$ otherwise. It is easy to verify that this is an SP, onto and unanimous mechanism; however, it is not 0-SI, since moving from the profile $(0, 1)$ to $(0, 2)$ changes the result from 0 to 2.

We now turn to prove the main theorem of this section.

Proof of Theorem 2. Suppose f is an onto SP mechanism; then, by Lemmas 1 and 3, it is also monotone and unanimous, and therefore, by Lemma 5, it is 1-SI. Suppose that f does not satisfy 1-SSI; then, the agent violating 1-SSI can benefit by moving away from the facility location (since, by monotonicity, the facility moves in the direction of the agent).

We now prove the other direction. Suppose f is an onto, monotone and 1-SSI mechanism. We will show that f is also SP. Suppose some agent j moves from a_j to $a'_j > a_j$. By monotonicity, $x' = f(a'_j, a_{-j}) \geq x = f(\mathbf{a})$. If $x \geq a_j$, then agent j does not benefit from the deviation. Otherwise, $x < a_j$; then, by 1-SI it holds that $x = a_j - 1$. By DI, it must hold that $|[a_j, a'_j] \cap [x, x']| \geq 2$, which means that $x' \geq a_j + 1$. Here again, agent j does not benefit from the deviation. □

4 SP mechanisms on a discrete cycle

sec:cycle

Schummer and Vohra prove that any onto SP mechanism on the continuous cycle must be a dictatorship. However, this is not true for discrete cycles. Clearly, any dictator mechanism is both unanimous and SP, but the converse does not hold.

Consider some cycle R_k of even length k, with any number of agents. The following is an example of an SP mechanism: The cycle is partitioned to $k/2$ pairs of neighboring points. Agent 1 decides on a pair. The location within this pair of points is decided by a majority vote of all other $n - 1$ agents. This is not a dictatorial mechanism, and in fact every agent can affect the outcome in some profiles.

Moreover, if the cycle is small, then there are even completely anonymous mechanisms (i.e., very far from dictatorships) that are SP. See Section 4.4 for detailed examples.

We still want to claim that when k is large enough, then any onto SP mechanism on the cycle R_k is "close" to a dictator. Note that even in the example above, the facility is always next to agent 1, which makes him a 1-dictator. Our second main result shows that this is always the case (see formal statement in Theorem 13).

Main theorem. *For sufficiently large cycles, any onto SP mechanism is* 1*-dictatorial.*

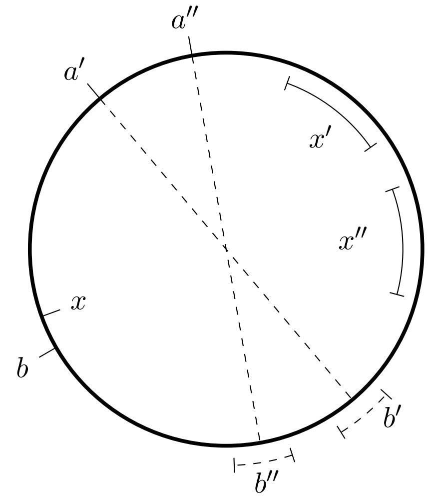
In Section 5 we complete the characterization (for even k) by considering the embedding of the cycle in the binary cube.

As a proof outline of the main theorem, we go through the following steps. We first consider the case of two agents, proving that any SP mechanism must be Pareto and then that the facility must always be next to one of the agents. It then follows that a 1-dictator must exist. The next step is proving the same for three agents, using a reduction to the $n = 2$ case (as in Schummer and Vohra). Finally, we extend the result to any number of agents. We believe that the outline of our proof can be easily followed and perhaps assist in the construction of similar characterization results.

Before diving into the case of 2 agents, we prove two general lemmas (a more formal statement is in Appendix C). Let $\mathbf{a}$, $\mathbf{a}'$ be two profiles that differ only by the location of one agent (w.l.o.g. agent 1), and denote $x = f(\mathbf{a}), x' = f(\mathbf{a}')$.

lemma:toward

Lemma 6. *If agent 1 moves* closer *to x along the shorter arc between them, then $x' = x$.*



rcle_two_100 Figure 1: Agents' locations (a, b, etc.) appear outside the cycle, and facility locations (x, x', etc.) appear inside.

lemma:jump **Lemma 7.** *Suppose that agent 1 moves one step* away *from x (along the longer arc between them), Let y be the point on the longer arc s.t. $d(a', y) = d(a, x)$. Then either $x' = x$ (no change); or $d(x', y) \leq 1$.*

For the case of the cycle we define a specific property of *cycle-Pareto*. For the exact definition see Def. 13 in the appendix. However, for our proof sketch it is sufficient to note that cycle-Pareto is very similar to Def. 3 (in fact for even size cycles the definitions coincide).

We prove below that any SP mechanism for 2 agents on large enough cycles must satisfy cycle-Pareto. However, as Lemma 20 in the appendix shows, this result is true for any number of agents.

4.1 2 agents on the cycle

ec:cycle_two
cycle_Pareto **Lemma 8.** *Let $k \geq 13$, $n = 2$. If f is SP and onto on R_k, then f is cycle-Pareto.*

Proof sketch of Lemma 8 for $k \geq 100$. We start from some profile where $x = f(a, b)$ is violating cycle-Pareto.

(1) We show (see Lemma 22 in Appendix C.1) that cycle-Pareto can only be violated when the facility is at distance (exactly) 2 from some agent (w.l.o.g. $d(b, x) = 2$), and agents are almost antipodal.

(2) We set two locations for agent 1, as $a' = x + 30 = b + 22$ and $a'' = x + 30 = b + 32$. Observe that $f(a', b) = f(a'', b) = x$. For each of the profiles a', b and a'', b, we move agent 2 counterclockwise (away from x), and denote by b' [respectively, b''] the first step s.t. $f(a', b') \neq x$ [resp., $f(a'', b'') \neq x$]. Denote $x' = f(a', b'), x'' = f(a'', b'')$. See Figure 1 for an illustration.

(3) We show that b' must be roughly antipodal to a', as otherwise we get a violating profile that is contradicting (1). Then by Lemma 7, x' is a reflection of x along the axis $a' \leftrightarrow b'$. It follows that[6]

$$|[x', x)| = |[x', b')| + |[b', x)| = 2|[b', x)| \pm 3 = k - 40 \pm 5.$$

(4) From a similar argument, we get that $|[x'', x)| = k - 60 \pm 5$, which means $x' \neq x''$.

[6] $A \pm B$ means "in the range $[A - B, A + B]$".

Finally, denote $z = f(a'', b')$. By Lemma 6, $z = f(a'', b'') = x''$, as agent 2 approaches x''. On the other hand, by the same argument $z = f(a', b') = x'$, as agent 1 approaches x'. Then we get a contradiction as $x' = z = x''$. □

h:dictator_2 **Theorem 9.** *Assume $k \geq 13$, $n = 2$. Let f be an onto SP mechanism on R_k, then f is a 1-dictator.*

The proof is using the cycle-Pareto property to show that the facility is always close to at least one agent. Then, we show that it is always the same agent. The full proofs of Lemma 8 and the theorem can be found in Appendix C.1.

4.2 3 agents on the cycle

:cycle_three

air_dictator **Lemma 10.** *Assume $k \geq 13$, $n = 3$. Let f be a unanimous SP mechanism on R_k. Then either f has a 1-dictator, or any pair is a 1-dictator. That is if there are two agents j, j' s.t. $a_j = a_{j'}$, then $d(f(\mathbf{a}), a_j) \leq 1$.*

ose_to_one_3 **Lemma 11.** *Let f be an SP, unanimous rule for 3 agents on R_k for $k \geq 13$. For all $a, b, c \in R_k$, $x = f(a, b, c)$, $d(a, x) \leq 1$ or $d(b, x) \leq 1$ or $d(c, x) \leq 1$.*

h:dictator_3 **Theorem 12.** *Assume $k \geq 22$, $n = 3$. Let f be an onto SP mechanism on R_k, then f is a 1-dictator.*

We give a simpler proof for large cycles. The full proof appears in Appendix C.2.

Proof for $k \geq 100$. Assume, toward a contradiction, that f has no 1-dictator. By Lemmas 11,10 we know that $f(\mathbf{a})$ is always close to at least one agent, and if there is a pair in the same place a^* then $d(f(\mathbf{a}), a^*) \leq 1$.

Let $\mathbf{a}$ be a profile where $a_2 = a_1 - 20$; $a_3 = a_1 + 20$. Thus all three agents and $x = f(\mathbf{a})$ are on the same semi-cycle and x is near a_1 (otherwise there is a manipulation for agent 2 or agent 3 by joining agent 1). Let $a'_2 = a_2 + 8$, then by Lemma 6, $f(a_1, a'_2, a_3) = x$. From each profile $\mathbf{a}, \mathbf{a}'$ we move agent 3 toward a_2 (or a'_2) along the *longer arc* between them, until the facility "jumps" to agent 2. This must occur at some point by Lemma 10. Denote by b_3 [resp., b'_3] the first point s.t. $f(a_1, a_2, b_3) \neq f(\mathbf{a})$ [resp., $f(a_1, a'_2, b'_3) \neq f(\mathbf{a}')$]. It must hold that b_3 is in the middle of the long arc between a_1, a_2 (plus or minus 1), since otherwise there would be a manipulation $b_3 \to b_3 - 1$ or vice versa. Thus $b_3 \in [a_1 + k/2 - 11, a_1 + k/2 - 9]$. From the same argument, $b'_3 \in [a_1 + k/2 - 7, a_1 + k/2 - 5]$ and therefore $b'_3 > b_3$. Finally, consider the two profiles $z = f(a_1, a'_2, b_3)$; $w = f(a_1, a_2, b_3)$. Since $b_3 < b'_3$, z is next to a_1, and thus $d(z, a'_2) \geq d(a_1, a'_2) - 1 = 11$. On the other hand, w is next to a_2 (by the definition of b_3), thus $d(w, a'_2) \leq d(a_2, a'_2) + 1 = 9 < 11 \leq d(z, a'_2)$, which means that $a'_2 \to a_2$ is a manipulation for agent 2, in contradiction to SP. □

4.3 n agents on the cycle

sec:cycle_n

Finally, we leverage the results of the previous sections to obtain a necessary condition for mechanisms on the discrete cycle for the general case of n agents. Proof is by induction and appears in Appendix C.3.

cle_dictator **Theorem 13.** *Let f be an onto and SP mechanism on R_k, where $k \geq 22$, then f is 1-dictatorial.*

The social cost While for a low number of agents these are not necessarily bad news (in fact, for $n = 2$ the dictator mechanism is optimal w.r.t. the social cost), for more agents this result provides us with a lower bound that linearly increases with the number of agents.

th:approx_n **Corollary 14.** *Every SP mechanism on R_k for $k \geq 22$ has an approximation ratio of at least $\frac{n}{2}$.*

4.4 Small cycles

:cycle_small

A natural question is the critical size of a cycle, for which there still exist SP mechanisms that are not 1-dictatorial. The proofs above show that the critical size for $n = 2$ is at most 12, and for $n \geq 3$ it is at most 21. We want to know whether these bounds are tight.

2_three_anon

Proposition 15. *There are onto and anonymous SP mechanisms for two and three agents on R_k, for all $k \leq 12$.*

For $k \leq 7$, the following "median-like" mechanism will work for $n = 3$: let $(a_3, a_1]$ be the longest clockwise arc between agents, then $f(\mathbf{a}) = a_2$. Break ties clockwise, if needed. For two agents we simply fix the location of one virtual agent (see proof in the appendix).

For $k \in [8, 12]$ the "median" mechanism is no longer SP, but we have been able to construct anonymous SP mechanisms using a computer search. A tabular description of these mechanisms is available online [1].[7]

We can now derive non-dictatorial SP-mechanisms for more than three agents by running the mechanism on an arbitrary pair of the agents, e.g., on the first two agents. Note however, that while such a mechanism is indeed not 1-dictator, it would not be anonymous.

5 The binary cube

sec:cube

A binary cube of dimension k is denoted by C_k. The set of vertices in C_k is the set of binary vectors of size k. Consider any set $V \subseteq C_k$. Two vertices $v, v' \in V$ are *connected* if their hamming distance (i.e., the number of coordinates in which they differ) is 1. Given a vertex v, we denote by $v[i] \in \{0, 1\}$ the i'th coordinate of v. It holds that if V forms a connected graph, then $d(v, v') = |\{i : v[i] \neq v'[i]\}|$.

We next define several properties of mechanisms for the binary cube C_k. These definitions will serve several purposes: first, by considering a natural embedding of R_k in C_k we can provide a full characterization of SP mechanisms on the cycle in terms of the cube dimensions. Interestingly, we give an alternative characterization for mechanisms on the line using the same properties. Second, we will consider some implications of our results on other domains, which correspond to the binary cube.

Suppose that V is some subset of C_k. Since every location can be thought of as having k coordinates (or attributes), the cube structure calls for some new definitions. We use the notation $a[i] \in \{0, 1\}$ for the i^{th} coordinate of a location a on the binary cube.

Definition 10. *A mechanism f is* cube-monotone *(CMON), if changing coordinate i of an agent can only change coordinate i in the same direction. That is, if $a_j[i] \neq a'_j[i]$ and $f(\mathbf{a})[i] \neq f(a_{-j}, a'_j)[i]$, then $f(\mathbf{a})[i] = a_j[i]$.*

Another property often considered in a multi-attribute setting is *independence in irrelevant attributes*. This means that coordinate i of the facility is only determined by the values of coordinate i of the agents' locations. While this property seems unnatural in the general case of aggregating agent location on a subset of the cube, it is reasonable in a lot of natural aggregation problems (E.g., in preference aggregation the IIA property means pair-wise aggregation and is accepted as a desired property. As was shown by [6] preference aggregation can be seen as aggregation on the cube). We relax this notion as follows.

Definition 11. *A mechanism f is m-*independent of irrelevant attributes *(m-IIA) if $f(\mathbf{a})[i]$ is determined by coordinates $i - m, \ldots, i + m$ of the voters in $\mathbf{a}$.*

[7] Since the number of mechanisms for 3 agents and bounded k is finite, we can in principle find the exact critical size by such a computer. However since the size of the search space is huge ($k^{\Theta(k^n)}$), exhaustive search is infeasible.

Note that the m-IIA property depends on coordinates order, and is not preserved under a permutation of coordinates' names. 0-IIA is just IIA. The following property is also quite natural.

def:IDA **Definition 12.** *A mechanism f is* independent of disjoint attributes *(IDA), if the coordinate changed by the agent and the coordinates changed in the facility (if it moved) always intersect. Formally, if a_j, a'_j differ by coordinates $S \subseteq K$, and $f(a_j, a_{-j}), f(a'_j, a_{-j})$ differ by coordinates $T \subseteq K$, then either $T = \emptyset$ (i.e. no change in outcome) or $S \cap T \neq \emptyset$.*

A similar property was suggested by Dietrich [7] as *independence in irrelevant information*(In our case a coordinate is relevant to its neighborhood).

We also say that a mechanism f is *Cube-Pareto*, if whenever all the agents agree on the same coordinate (vote the same), then this is the aggregated coordinate as well.

5.1 Embedding the line in the binary cube

We give a natural embedding of L_k in C_k. We map every $x \in L_k = \{0, 1, \dots, k\}$ to a vector $\varphi(x) \in \{0,1\}^k$, whose first x entries are 1. Thus $\varphi(x)[i] = 1$ iff $i \leq x$. It is easy to verify that g is distance-preserving, i.e., that $d(\varphi(x), \varphi(x')) = |x - x'| = d(x, x')$.

It is also not hard to see that properties of mechanism on L_k can be mapped to other properties of mechanisms over $\varphi(L_k) \subseteq C_k$ (see Lemma 24). This mapping gives us the following theorem, which is equivalent to Theorem 2.

th:line_cube **Theorem 16.** *An onto mechanism on the line L_k (embedded in C_k) is SP if and only if it is 1-IIA, CMON, and IDA.*

5.2 Full characterization of SP mechanism on the cycle

Every cycle of even length can be thought of as "two lines attached in their ends". Indeed, R_{2k} can be embedded in the binary cube C_k in a very similar way to the embedding of the line. This is by mapping the first k points on the cycle (setting order and orientation on the cycle. We later show that these can be arbitrarily chosen) to vectors of the form $0^{k_1}1^{k_2}$ (as with L_k), and the remaining k points to vectors of the form $1^{k_1}0^{k_2}$. In particular, $\varphi(0) = 0^k$, and $\varphi(k) = 1^k$. As with L_k, it is not hard to verify that our mapping preserves distances, as $d(\varphi(x), \varphi(x')) = d(x, x') = |x - x'| \pmod{2k}$.

We can now turn to completing the characterization SP mechanisms on the cycle, extending Theorem 13.

:circle_char **Theorem 17.** *Let $2k > 13$. An onto mechanism on the cycle R_{2k} is SP if and only if it is 1-dictatorial, CMON, and IDA.*

Notice that all these properties do not depend on the choice of embedding (from the $2k$ ways to choose starting point and direction). The coordinates can be seen as a geometric property telling us where is the facility w.r.t. the agents and their antipodal points, hence the properties are independent of the embedding. For instance, Cube-Pareto can be interpreted as - "for any semi-cycle s.t. all the agents are in this semi-cycle, the facility should lie in it as well".

6 Discussion

c:discussion

Our two primary results are the complete characterization of onto SP mechanisms on the discrete line, and proving that on sufficiently large cycles, every onto SP mechanism must be close to a dictatorship. We further studied how this characterization is affected by the cycle size and the number of agents (see Table 1), and completed the characterization of onto SP mechanisms for even sized cycles in these cases where the dictatorial condition holds.

	$k \leq 12$	$k \in [13, 21]$	$k \geq 22$
$n = 2$	**A** $\Uparrow$	**D** (Th. 9)	
$n = 3$	**A** (Prop.15)	?	**D** (Th. 12)
$n > 3$	**ND** $\Downarrow$		**D** (Th. 13)

tab:results Table 1: Summary of results for SP mechanisms on R_k, with n agents. **D** means that every SP mechanism is 1-dictatorial. **ND** means there exists an SP non-1-dictatorial mechanism. **A** means there exists an SP anonymous mechanism. We conjecture that $k = 12$ is the critical value for the characterization to hold for every $n \geq 2$.

Implications on other domains We mentioned in the introduction that there is a mapping between facility location mechanisms in our model, and binary classification mechanisms operating on realizable datasets. A natural question is whether we can derive characterization and approximation results that will apply for the classification setting, and in particular to natural concept classes that are in use in the machine learning literature. A corollary of Theorem 13 is that any SP classification mechanism has an approximation ratio of $\Omega(n)$. The result holds even when restricted to realizable datasets, and it is therefore stronger than the result of Meir et al. [13]. For more details see Appendix E.

Future directions Obvious future directions include proving the conjecture left in the paper regarding the exact size of a cycle that entails near-dictatorships. We also conjecture that the characterization of line mechanisms can be extended to trees, similarly to the result from Schummer and Vohra [21]. Other directions include the characterization of SP mechanisms (both deterministic and randomized) and the study of their approximation bounds for a variety of topologies and optimization criteria. An intriguing open question is whether randomized SP mechanisms (on a particular structure) must also be close to (random) dictatorship, as we already know to be true for a more general model [15].

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A Appendix for Section 2

reliminaries

Lemma 1. *Every mechanism f that is both onto and SP, is unanimous.*

Proof. Suppose toward contradiction that $f(a, a, \ldots, a) = b$ for some $a \neq b$. Since f is onto, there exists a profile $\mathbf{c}$ such that $f(\mathbf{c}) = a$. We iteratively switch agents' locations from $a_j = a$ to c_j, and denote by j^* the first agent such that $f(\{c_j\}_{j \leq j^*}, \{a_j\}_{j>j^*}) = a$. The existence of such an agent is guaranteed since $f(\mathbf{c}) = a$. It follows that agent j^* can benefit by misreporting c_{j^*} instead of a in the profile $(\{c_j\}_{j \leq j^*-1}, \{a_j\}_{j>j^*-1})$, contradicting that f is SP. The assertion of the lemma follows. □

B Appendix for Section 3

apx:line

Lemma 3. *Every SP mechanism is monotone.*

Proof. Suppose that f is SP, and assume toward a contradiction that f is not monotone. Thus there are $\mathbf{a} \in V^n$, $j \in N$, $b_j > a_j$ such that $x' = f(a_{-j}, b_j) < f(\mathbf{a}) = x$. We have that $|x' - a_j| \geq |x - a_j|$, since otherwise j can benefit by reporting b_j in the profile $\mathbf{a}$. Since $x' < x$, this implies $x' < a_j < b_j$. Thus $|x - b_j| < |x' - b_j|$, and thus j can benefit by reporting a_j instead of b_j. □

Lemma 4. *A monotone mechanism f is Pareto iff it is unanimous.*

Proof. Pareto clearly entails unanimity. It remains to show that unanimity implies Pareto. Note that the notion of Pareto on a line is equivalent to $f(\mathbf{a}) \in [\min_{j \in N} a_j, \max_{j \in N} a_j]$. Let $\mathbf{a} \in L^n$, $a' = \min_{j \in N} a_j$ and $a'' = \max_{j \in N} a_j$. Also let $\mathbf{a}' = (a', \ldots, a')$ and $\mathbf{a}'' = (a'', \ldots, a'')$. By unanimity, $f(\mathbf{a}') = a'$ and $f(\mathbf{a}'') = a''$. By monotonicity,

$$\min_{j \in N} a_j = a' = f(\mathbf{a}') \leq f(\mathbf{a}) \leq f(\mathbf{a}'') = a'' = \max_{j \in N} a_j,$$

as required. □

Lemma 5. *Every SP, unanimous mechanism for the line is* 1*-SI.*

Proof. Since f is SP and unanimous, it is MON (by Lemma 3) and thus also Pareto (by Lemma 4). Suppose by way of contradiction that f is not 1-SI. Then, there exists at least one pair of profiles that violates the 1-SI property. Assume w.l.o.g. that the two profiles differ only in agent 1's report, and also that $a_1' = a_1 + 1$. Among these pairs, let $\mathbf{a}, \mathbf{a}'$ be the two profiles that minimize $\sum_{j \in N} a_j$. By the violation of the 1-SI property, it holds that $f(a_1, a_{-1}) = x \neq x' = f(a_1', a_{-1})$, while $d([a, a'], f(\mathbf{a})) > 1$. Since $a_1' > a_1$, MON implies that $x' > x$. Let $I_a = [a_1, a_1']$ and $I_x = [x, x']$.

We claim, in addition, that w.l.o.g. agent 1 is the extreme minimal agent, i.e.

$$a_1 \leq a_j \text{ for every } j \in N. \quad (1)$$

eq:a1_min

To see this, suppose that there exists some agent, w.l.o.g. agent 2, such that $a_2 = \min_{j \in N} a_j < a_1$. We define two new profiles, $\mathbf{b}, \mathbf{b}'$, that differ from $\mathbf{a}, \mathbf{a}'$ only by relocating agent 2 so that $b_2 = b_2' = a_1'$. Denote the new output locations by $y = f(\mathbf{b})$ and $y' = f(\mathbf{b}')$. One can easily verify that $b_2 \geq a_2$ and $b_2' \geq a_2'$. It, therefore, follows from MON that $y \geq x$, $y' \geq x'$, and also $y' \geq y$. We distinguish between two cases.

If $y' > y$, then, since $y' \geq x' \geq a_1'$, the profiles $\mathbf{b}, \mathbf{b}'$ violate the 1-SI property. Moreover, since $b_2 = a_1' > a_1 > a_2$, we get that

$$\sum_{j \in N} b_j = \sum_{j \neq 2} a_j + b_2 = \sum_{j \neq 2} a_j + a_1' > \sum_{j \in N} a_j,$$

in contradiction to the maximality of the violating pair $\mathbf{a}, \mathbf{a}'$.

If $y = y'$, then consider the pair of profiles $\mathbf{a}, \mathbf{b}$ and their outcomes $x = f(\mathbf{a}), y = f(\mathbf{b})$. It holds that $x < x' \leq y' = y$ and $y = y' \geq x' \geq a'_1 = b_1$. This means that agent 2 affects coordinate $i > b_2+1$ by moving from a_2 to b_2. It follows that the pair $\mathbf{a}, \mathbf{b}$ violate the 1-SI property. As this pair is still maximal with respect to the sum of the agents' locations, we rename the agents, swapping agents 1 and 2, and find a single step violating 1-SI in the interval $[a_2, b_2]$. This establishes the assertion of the claim.

We are now ready to prove the lemma; we distinguish between the following cases:

case a: The intervals I_a, I_x intersect on at most one point. It follows that either $a_1 \geq x'$ or $a'_1 \leq x$. In the former case, agent 1 can benefit by reporting a'_1 instead of a_1 in $\mathbf{a}$, and similarly, in the latter case, agent 1 can benefit by reporting a_1 instead of a'_1 in $\mathbf{a}'$. Thus, a contradiction is reached.

case b: One of the intervals I_a, I_x strictly contains the other. Since $a'_1 = a_1 + 1$ and $f(a_1, a_{-1}) \neq f(a'_1, a_{-1})$, the inclusion must be $I_a \subsetneq I_x$. This implies that $x < a_1$. By Equation 1, it follows that $x < a_j$ for every $j \in N$, in contradiction to Pareto. □

ma:IIA_to_SP **Lemma 18.** *Every* 0*-SI, MON mechanism f for the line is SP.*

Proof. Let $\mathbf{a}$ be a profile, j an agent, and a'_j a deviation of j and assume that $f(a_j, a_{-j}) \neq f(a'_j, a_{-j})$. W.l.o.g., assume $a_j < a'_j$. From monotonicity we get that $f(a_j, a_{-j}) < f(a'_j, a_{-j})$ and from f being 0-SI $a_j \leq f(a_j, a_{-j}) < f(a'_j, a_{-j}) \leq a'_j$. Therefore, we get that j prefers $f(a_j, a_{-j})$ in $\mathbf{a}$ and hence this is not a manipulation. □

C Appendix for Section 4

apx:cycle

Let $\mathbf{a}$, $\mathbf{a}'$ be two profiles that differ only by the location of one agent (w.l.o.g. agent 1), and denote $x = f(\mathbf{a}), x' = f(\mathbf{a}')$. **Lemma 6.** *If agent 1 moves* closer *to x along the shorter arc between them, then $x' = x$. I.e., if $|(a, x]| \leq \lfloor k/2 \rfloor$ and $a' \in (a, x]$ then $x' = x$.*

Proof. W.l.o.g. we assume that a moves clockwise, and prove by induction on the number of steps toward x. Let $a' = a + 1$. Assume, toward a contradiction, that $y = f(a', a_{-1}) \neq x$. Then either $y \in [a, x)$ (in which case $a \to a'$ is a manipulation), or $y \in (x, a)$. If $|[a, y]| \leq \lfloor k/2 \rfloor$, then since $x \in (a', y)$ it is closer to $a' = a + 1$ than y, meaning that $a' \to a$ is a manipulation.

Therefore, the shorter arc between a, y is $[y, a]$, of length $\leq \lfloor k/2 \rfloor$. Of course, $d(y, a) \geq d(x, a)$ (otherwise $a \to a'$) is a manipulation. However, this means that

$$d(a', x) = d(a, x) - 1 \leq d(a, y) - 1 = (d(a', y) - 1) - 1 < d(a', y),$$

i.e.,that $a' \to a$ is a manipulation. □

Lemma 7. *Suppose that agent 1 moves one step* away *from x (along the longer arc between them), Let y be the point on the longer arc s.t. $d(a', y) = d(a, x)$. Then either $x' = x$ (no change); or $d(x', y) \leq 1$.*
(If x is antipodal to a, then trivially a cannot move it.)

Proof. W.l.o.g. $a' = a + 1$. Denote $x' = f(a', b)$. If $x' \in (x, y - 2]$ then $a \to a'$ is a manipulation. If $x' \in [y + 2, x)$ then $d(a', x') > d(a', x)$, and thus $a \to a'$ is a manipulation. □

cycle_Pareto **Definition 13.** *Let $a_1, \ldots, a_n$, s.t. the minimal arc (consecutive part of the cycle) that includes all the points is of size $< k/2$.*[8] *A point $x \in R_k$ is* cycle-Pareto *(w.r.t. the profile a_i), in either of the following cases:*

[8] Notice that the minimal arc is uniquely defined in such case.

- *x lies on the arc.*
- *k is odd, the arc size is $\lfloor k/2 \rfloor$, and there is an agent i next to x, i.e., $d(a_i, x) = 1$.*

If there is no such arc, every point $x \in R_k$ is cycle-Pareto.
A mechanism f is cycle-Pareto*, if for any profile $x = f(a_1, a_2, \ldots, a_n)$ is a cycle-Pareto outcome.*

reto_between **Lemma 19.** *For an odd cycle, a profile $(a_1, \ldots, a_n)$, and a point x:*

i *If x is a cycle-Pareto outcome, then there is no point y s.t. $d(a_i, y) < d(a_i, x)$ for every agent i*[9]

ii *If x is not a cycle-Pareto outcome, then there exists a point y s.t. $d(a_i, y) \le d(a_i, x)$ for every agent i and $d(a_{i'}, y) < d(a_{i'}, x)$ for some agent i'*[10].

Proof.

i Clearly, if x is a cycle-Pareto outcome due to the first condition (lies on the arc), it is also Pareto.
Otherwise, x does not lie on the arc, the arc size is exactly $\lfloor k/2 \rfloor$, and there is an agent s.t. $d(a_i, x) = 1$. W.l.o.g, the agents are ordered clockwise $a_1, a_2, \ldots, a_n$ s.t. $d(a_1, a_n) = \frac{k-1}{2}$, $d(x, a_1) = 1$, $d(x, a_n) = \frac{k-1}{2}$. The only point that is closer to a_1 than x is a_1 itself and it is not closer to a_n than x.

ii If x is not cycle-Pareto then the minimal arc is of size $\le (k-1)/2$ and x does not lie on this arc. W.l.o.g, the agents are ordered clockwise $a_1, a_2, \ldots, a_n$ and a_1 is the closest to x. I.e., $d(a_1, x) \le d(a_i, x)$ for all i. Denote by $t = d(x, a_1) > 0$.
If $t + d(a_1, a_n) \le (k-1)/2$ (x and all the points lie on a semi-cycle), then for all agents $d(a_1, a_i) \le d(x, a_i)$ and $d(a_1, a_1) = 0 < d(x, a_1)$ so $y = a_1$ satisfies the conditions.
If $t > d(a_1, a_n)$, then for all agents $d(a_i, a_n) \le d(a_1, a_n) < d(a_1, x)$ so
$d(x, a_i) = \min\left(d\left(a_i, a_1\right) + d\left(a_1, x\right), d\left(a_i, a_n\right) + d\left(a_n, x\right)\right) \ge \min\left(d\left(a_1, x\right), d\left(a_i, a_n\right)\right) = d\left(a_i, a_n\right)$ and $d(a_n, a_n) = 0 < d(x, a_n)$ so $y = a_n$ satisfies the conditions.
Otherwise, the point y is defined as the point on the arc $[a_1, a_n]$ s.t. $d(a_1, y) = t - 1$. For any agent: If $d\left(a_i, a_1\right) + d\left(a_1, x\right) < k/2$ then $d\left(a_i, y\right) < d\left(a_i, x\right)$. Otherwise, $d\left(a_i, y\right) - d\left(a_i, x\right) = \left(d\left(a_i, a_1\right) - d\left(a_1, x\right) + 1\right) - \left(k - d\left(a_i, a_1\right) - d\left(a_1, x\right)\right) = 2d\left(a_i, a_1\right) - k + 1 \le 0$ so y satisfies the conditions. □

mma:Pareto_n **Lemma 20.** *If f is an SP mechanism for R_k for $n > 2$ agents that does not satisfy cycle-Pareto then there exists an SP mechanism g for R_k for 2 agents that does not satisfy cycle-Pareto.*

Proof. Let $(a_1, \ldots, a_n)$ be a profile s.t. $x = f(a_1, \ldots, a_n)$ is not a cycle-Pareto outcome. So we know that all the points lie on an arc smaller than $k/2$. W.l.o.g, assume a_1, a_2 are the extreme points of this arc. We define g by $g(u, v) = f(u, v, a_3, \ldots, a_n)$.
Since f is SP, so is g and clearly $x = g(a_1, a_2)$ is not a cycle-Pareto outcome. □

C.1 Appendix for Sub-Section 4.1

px:cycle_two
lemma:pareto **Lemma 21.** *If $a, b, f(a, b)$ are on the same semi-cycle, then $f(a, b)$ must be between a, b.*

Proof. Assume otherwise, w.l.o.g. $a \in (b, f(a, b))$. Then b can manipulate by reporting a, since $f(a, a)$ is closer to b than $f(a, b)$. □

[9] In the literature this criterion is usually referred to as 'Strong Pareto Dominance' and is equivalent to definition 3.
[10] In the literature this criterion is usually referred to as 'Weak Pareto Dominance'.

to_antipodal **Lemma 22.** *Let f be onto and SP rule on R_k ($k \geq 13$). Suppose that $x = f(a,b)$ is violating cycle-Pareto. Then x is at distance (exactly) 2 from some agent, and agents are almost antipodal, i.e. $k/2 > d(a,b) \geq k/2 - 1$.*

Proof. Let $f(a,b) = x$ such that x is not cycle-Pareto. Moreover, let a, b be the profile minimizing $d(a,b)$ under this condition. W.l.o.g. $[a,b]$ is the shorter arc (we denote $a < b$), thus $x \in (b,a)$. By unanimity, $d(a,b) \geq 5$, as otherwise there is a manipulation $a \to b$ or vice versa (as either $d(a,x) > 4$ or $d(b,x) > 4$). We denote $u = f(a+1,b)$ and $w = f(a,b-1)$. By minimality of $d(a,b)$, $u, w \neq x$ and both u, w are cycle-Pareto (w.r.t. their respective profiles). See Figure 2 for an illustration. We prove the following series of claims.

- $u \neq w$ (W.l.o.g. that $u = b$)
  Indeed, suppose that they are equal, then $u = w \in [a,b]$, and thus $d(a,u) + d(b,u) = d(a,b)$. Also, from SP we have that $d(a,x) \leq d(a,u)$ and $d(b,x) \leq d(b,u)$. By joining the inequalities,

  $$d(a,b) = d(a,u) + d(b,u) \geq d(a,x) + d(b,x) = k - d(a,b).$$

  This entails that $[a,b]$ is the long arc, which is a contradiction.

- Either $w = a$ or $u = b$.
  Consider $q = f(a+1, b-1)$. If $u \neq b$, then $q = f(a+1,b-1) = f(a+1,b) = u$ by Lemma 6. Similarly, if $w \neq a$, then $q = f(a+1,b-1) = f(a,b-1) = w$. Therefore if neither of the two equalities holds then $u = q = w$ in contradiction to the previous claim.

- $d(q, b-1) \leq 1$
  Otherwise $b - 1 \to b$ is a manipulation for agent 2 (under $a + 1$).

- $d(q,b) = d(q,b-1) + 1 \leq 2$.

- $a \neq w$
  If $a = w$, then $d(q, a+1) \leq 1$, i.e. $q = a+1$ or $q = a+2$, and thus (since $k \geq 13$) $d(q,b) = d(a,b) - d(q,a) \geq 5 - 2 = 3$, in contradiction to the previous result.

- $1 \leq d(x,b) \leq 2$
  Deviation of agent 1 $a \to a+1$ is not beneficial (under b) and hence $d(x,b) \leq d(w,b) = d(q,b) \leq 2$

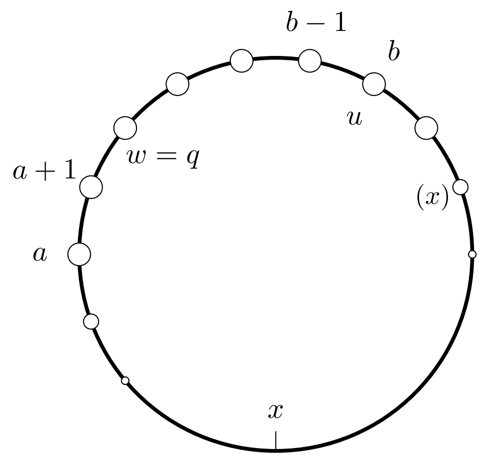
- $d(x,b) = 2$
  If k is even, then $d(a,b) < k$, as otherwise every outcome is Pareto. It then follows by Lemma 21 that $d(x,b) > 1$ (i.e. $d(x,b) = 2$). Similarly, if k is odd, then $d(a,b) \leq \lfloor k/2 \rfloor$ (otherwise $a \to b$ is a manipulation). Then $d(b,x) = 2$ as well, since $d(a,x) = 1$ would not violate cycle-Pareto by definition. Thus we get the first part of the lemma.

- $d(a,b) \geq k/2 - 1$. Suppose otherwise, i.e. that $d(a,b) < k/2 - 1$. Since $d(x,b) = 2$, then $d(a,x) > d(a,b)$, and $a \to b$ is a manipulation for agent 1.

□

Lemma 8. *Let $k \geq 13$, $n = 2$. If f is SP and onto on R_k, then f is cycle-Pareto.*

Proof of Lemma 8 for $k \geq 13$. Recall that by Lemma 22 cycle-Pareto can only be violated when the facility is at distance (exactly) 2 from some agent, and agents are almost antipodal.



rcle_two_uwq

Figure 2: An illustration of the original violating profile, where $x = f(a, b)$. Agents' locations (a, b, etc.) appear outside the cycle, and facility locations (x, x', etc.) appear inside. The conclusion of this part of the proof is that the facility must be close to one of the agents (appears in brackets).

- For any location of agent 1 on $[x, a] = \{b + 2, b + 3, \ldots, a\}$, the location of the facility does not change (by Lemma 6). In particular, this includes the locations $a' = x + 3 = b + 5$ and $a'' = x + 5 = b + 7$ (since $k \geq 13$ it holds that $a', a'' \in [x, a]$).

  We claim that the existence of profiles (a', b) and (a'', b) leads to a contradiction. This claim completes our proof, and will also be used in the proof of Lemma 23.

  For each of the profiles a', b and a'', b, we move agent 2 counterclockwise (away from x), and denote by b' [respectively, b''] the first step s.t. $f(a', b') \neq x$ [resp., $f(a'', b'') \neq x$]. See Figure 3(a) for an illustration.

- $b'' > b'$

  Indeed, the facility cannot move before agent 2 crosses the point antipodal to a'' (i.e. while $|[b - t, a'']| < |[a'', b - t]|$), as otherwise we would have a profile violating cycle-Pareto, where $d(x, a''), d(x, b - t) \neq 2$. Similarly, the facility *must* move after the crossing (i.e. when $|[b - t, a'']| > |[a'', b - t]|$). Thus for odd k we get that $b'' = a'' - \lceil k/2 \rceil$. For even k, the location of the facility can be anywhere when a'', b'' are exactly antipodal, thus $b'' \in [a'' - k/2 - 1, a'' - k/2]$.

- We can summarize both cases with the following constraint:

$$d(b'', x) = |(b'', x]| = |(b'', a'']| - 5 = \left(\frac{k}{2} + \frac{1}{2} \pm \frac{1}{2}\right) - 5 \leq \frac{k}{2} - 4. \tag{2}$$

eq:d_b2_x

  A similar analysis for a', b' shows the following:

$$d(b', x) = |(b', x]| = |(b', a']| - 3 = \left(\frac{k}{2} + \frac{1}{2} \pm \frac{1}{2}\right) - 3 \geq \frac{k}{2} - 3. \tag{3}$$

eq:d_b1_x

- We denote $x' = f(a', b'), x'' = f(a'', b'')$. By Lemma 7, x' is roughly the same distance from b' as x is, i.e.

$$d(b'', x'') = d(b'' - 1, x) \pm 1 \leq d(b'', x) \leq k/2 - 4 \qquad \text{(By Eq. (2))}$$

eq:d_b2_x2

$$d(b', x') = d(b' - 1, x) \pm 1 \geq d(b', x) - 2 \geq k/2 - 5 \qquad \text{(By Eq. (3))}$$

eq:d_b1_x1

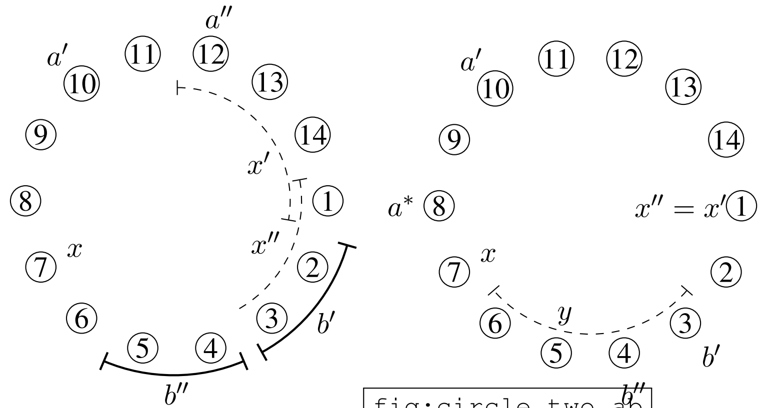


fig:circle_two_ab fig:circle_two_xy

(a) The ranges of the critical locations of agent 2 b', b'' are shown, as well as the ranges of the outcomes $x' = f(a', b'), x'' = f(a'', b'')$.

(b) An illustration of case (ii).

Figure 3: Illustrations of profiles defined in the second part of the proof.

We get that either (i) x'' is strictly closer that x' to b' (i.e. $x'' < x'$), or (ii) $x'' = x', b'' = b' + 1$. Also note that by Lemma 7 $d(b', x') \leq d(b' - 1, x)$, which entails (for $k \geq 13$) that $x' \geq a''$. See Figure 3(a) for an illustration.

(i) Suppose that $x'' > x'$. This case leads to a simple contradiction. Denote $z = f(a'', b')$. By Lemma 6, $z = f(a'', b'') = x''$. Similarly, since $x' \geq a'$, $z = f(a', b') = x'$, in contradiction to the assumption.

(ii) Suppose that $x' = x''$, $b' = b'' - 1$ (see Figure 3(b)). In particular it must hold that k is even and $d(a', b') = k/2$, i.e. $k \geq 14$. We move agent 1 two steps from x' counter clockwise, to $x + 1$. We denote this location of agent 1 by a^*. Since $b'' > b'$, $f(a', b'') = x$ and thus by Lemma 6 $f(a^*, b'') = x$ as well.

We argue that $y = f(a^*, b') \neq x'$. Suppose otherwise, then $y = x'$ is violating cycle-Pareto. Moreover, $d(a^*, b') < k/2 - 1 \leq d(a, b)$ in contradiction to minimality. It thus follows that $y \in [b', a^*]$. We further argue that $y \in [b', x)$. Indeed, if $y \in [x, a^*]$, then $d(a', y) \leq 3$. On the other hand,

$$d(a', x') = d(a', b') - d(b', x') = k/2 - (k/2 - 5) = 5,$$

thus $a' \to a^*$ is a manipulation for agent 1 under b'.

- Finally, it follows that agent 2 has a manipulation by moving from b'' to b' under a^*. This is since the facility moves from $x = f(a^*, b'')$ (where $d(x, b'') \geq 2$) to $y = f(a^*, b')$, where $y \in (x, b']$ and thus $d(y, b'') < d(x, b'')$. □

Finally, due to Lemma 20, SP mechanisms on R_k, $k \geq 13$ must be cycle-Pareto, for every n.

close_to_one **Lemma 23.** *Let $k \geq 13$. For all $a, b \in R_k$, $x = f(a, b)$, $d(a, x) \leq 1$ or $d(b, x) \leq 1$.*

Proof. Assume that there is some violating profile, then w.l.o.g. it is $x = f(a, b)$, where $x = b + 2$, and $a > x$. Also, by cycle-Pareto and Lemma 6, $a' = b + 5$,$a'' = b + 7$ have the same outcome $x = f(a', b) = f(a'', b)$.

However, we explicitly claimed in the proof of Lemma 8 that such a profile leads to a contradiction. □

Theorem 9. *Assume $k \geq 13$, $n = 2$. Let f be an onto SP mechanism on R_k, then f is a 1-dictator.*

Proof. Take some profile a, b where $x = f(a, b)$, $d(x, b) > 1$. By Lemma 23, x is near a, i.e., $d(a, x) \leq 1$. We will show that agent 1 is a 1-dictator (in the symmetric case, agent 2 will be a 1-dictator). Assume, toward a contradiction, that there is some location b' for agent 2 s.t. $d(y, a) > 1$, where $y = f(a, b')$ (and by Lemma 23, $d(y, b') \leq 1$). We can gradually move agent 2 from b to b' until the change occurs, and thus w.l.o.g. $b' = b + 1$. By Lemma 6, moving agent 2 toward x cannot change the outcome, thus on the arc $[x, b']$ the order is $x < x + 1 < b < b'$.

We must have $d(b, x) \leq d(b, y)$ otherwise there is a manipulation $b \to b'$. Thus $d(b, a) - 1 \leq d(b, b') + 1 = 2$, i.e., $d(a, b) \leq 3$. Also, $d(a, b) \geq 1$ since otherwise $d(y, a) = d(y, b) \leq 1$ in contradiction to our assumption. Thus there are three possible cases, and we will show that each leads a contradiction.

(I) If $d(a, b) = 1$, then since $d(x, b) > 1$ we have $x = a - 1$. However this contradicts lemma 21.

(II) If $d(a, b) = 2$, then $x = a$ (since $x = a - 1$ contradicts lemma 21). Thus $d(y, b) \geq d(x, b) = d(a, b) = 2$ which means $y = b' + 1 = b + 2$. This induces a manipulation for agent 1 $a \to b'$ (by unanimity).

(III) If $d(a, b) = 3$, then since $k > 8$, all of the points are on a semi-cycle and thus $x \in \{a, a+1\}$, $y \in \{b', b' - 1\}$ (again, by lemma 21). However this clearly means that $d(y, b) \leq 1 < d(x, b)$, and there is a manipulation $b \to b'$ for agent 2. □

C.2 Appendix for Sub-Section 4.2

:cycle_three

Lemma 10. *Assume $k \geq 13$, $n = 3$. Let f be a unanimous SP mechanism on R_k. Then either f has a 1-dictator, or any pair is a 1-dictator. That is if there are two agents j, j' s.t. $a_j = a_{j'}$, then $d(f(\mathbf{a}), a_j) \leq 1$.*

Proof. Let f be an SP unanimous rule for $n = 3$ agents. We define a two agent mechanism for every pair $j, j' \in N$ by letting j be a duplicate of j' (For ease of notation we'll refer to the agents of $g^{j,j'}$ by agent I and agent II, the third agent by j'', and the original agents by agent 1, agent 2, and agent 3),

$$g^{12}(a,b) = f(a,a,b)$$
$$g^{23}(a,b) = f(b,a,a)$$
$$g^{31}(a,b) = f(a,b,a).$$

Clearly, the mechanism $g^{j,j'}$ is unanimous, since $\varphi(a, a) = f(a, a, a) = a$.

We argue that $g^{j,j'}$ is SP. Indeed, otherwise there is a manipulation either for agent II (which is also a manipulation in f, which is a contradiction to SP) or for agent I (say, $a \to a'$). In the latter case we can construct a manipulation in f by iteratively switching agents j, j' from a to a'. Either j or j' strictly gains by this move and thus has a manipulation.

Since $g^{j,j'}$ is a unanimous and SP, by Theorem 9 it has a 1-dictator. If the dictator is agent II then j'' is also a 1-dictator of f. Otherwise, suppose that $f(a_j, a_{j'}, a_{j''}) = x$, and $d(x, a_{j''}) > 1$. However It the follows by Lemma 6 that $f(x, x, a_{j''}) = x$ as well, which is a contradiction.

If agent I is a 1-dictator of g, then whenever $a_j = a_{j'}$, $d(f(\mathbf{a}), a_j) \leq 1$. □

Lemma 11. *Let f be an SP, unanimous rule for 3 agents on R_k for $k \geq 13$. For all $a, b, c \in R_k$, $x = f(a, b, c)$, $d(a, x) \leq 1$ or $d(b, x) \leq 1$ or $d(c, x) \leq 1$.*

Proof. By Lemma 10, either there is a 1-dictator (in which case we are done), or every pair of agents standing together serve as a 1-dictator.

Let $u_1, u_2, u_3, x = f(u_1, u_2, u_3)$ s.t. x is at least 2 steps from all agents. We have that there is a semi-cycle in which x and two other points are consequent, and thus x must be between them

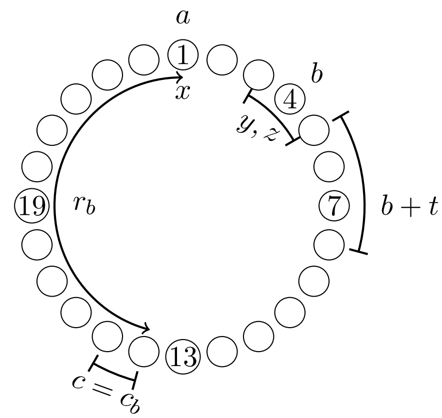


Figure 4: An illustration of the profiles on R_{24}. Note that $c = c_b$ must be roughly half way between a an b, and also roughly half way between a and $b + t$.

circle_three

(otherwise the more distant agent of the two has a manipulation, similarly to Lemma 21). W.l.o.g. $u_1 + 1 < x < u_2 - 1$ (i.e., ordered that way on an arc). Now suppose that agent 3 moves to u_1 or u_2, whichever closer to her (assume u_1). Then $y = f(u_1, u_2, u_1)$ is close to u_1. We thus have

$$d(u_3, y) \leq d(u_3, u_1) + d(u_1, y) \leq (d(u_3, x) - 2) + 1 < d(u_3, x),$$

i.e.,there is a manipulation for agent 3. □

Theorem 12. *Assume $k \geq 22$, $n = 3$. Let f be an onto SP mechanism on R_k, then f is a 1-dictator.*

Proof for $k \geq 22$. Assume, toward a contradiction, that there is no 1-dictator.

We begin from a profile where $a = 1$, $b'' = 6$. Let $x = f(a, b'', a)$. By Lemma 10, $d(x, a) \leq 1$. Moreover, we can assume w.l.o.g. that $x = a = 1$, since otherwise we can move all agents toward x.

We also define an alternative profile, where $b' = b'' + 3 = 9$.

We now move agent 3 counterclockwise from a to b' (i.e., along the long arc). Suppose $c_0 = a - 3$. Then the facility is either near a or near c_0. If it is near c_0, then it must follow agent 3 all the way to b' (otherwise he will have a manipulation). In particular, the facility must visit one of the points $b' + 2, b' + 3, b' + 4$. However if $k^* \geq 22$, then all three points are at distance of at least 10 from a. This means that agent 1 can manipulate $a \to b'$, bringing the facility (by Lemma 10) to $b' \pm 1$.

Therefore, $f(a, b', c)$ is either near a or near b' for all $c \in [b', a]$. Moreover, when it is near a we can still assume it equals $x = a$ (since otherwise we again move agents 1 and 2 toward the facility).

We can now apply exactly the same argument on the initial profile (a, b'', c_0), thus $f(a, b'', c)$ is either near a or near b'' for all $c \in [b'', a]$.

Next, let c'_b denote the last point on the long arc s.t. $f(a, b', c_{b'}) = x$, meaning that $d(z, b') \leq 1$, where $z = f(a, b', c)$ for all $c \in (b, c_{b'+1}]$ (see Figure 4). Similarly, $c_{b''}$ will denote the critical switching point when agent 2 is at b''.

Our plan is as follows: First to show that $c_{b'}, c_{b''}$ are roughly on the middle of the long arc $[b', a]$ and $[b'', a]$, respectively. It will then follow that $c_{b'} > c_{b''}$, which will lead to a manipulation of agent 2.

Let us study the constraints on c_b (for any location b of agent 2). For convenience, we denote $r_b = k - c_b = d(c_b, a)$. Indeed, $r_b = d(x, c_b) \leq d(z, c_b)$, otherwise there is a manipulation $c_b \to c_b + 1$ for agent 3. Thus,

$$r_b = d(x, c_b) \leq d(z, c_b) \leq d(b, c_b) + d(b, z) \leq d(b, c_b) + 1 = k - b - r_b + 1,$$

i.e.,$r_b \leq \frac{k-b+1}{2}$.

Similarly, $d(c_b+1,z) \leq d(c_b+1,x) = r_b+1$ (otherwise $c_b+1 \to c_b$ is a manipulation), thus $r_b \geq \frac{k-b-3}{2}$.

Substituting b for the actual values b', b'', we get that

$$c_{b'} = k - r_{b'} \geq k - \frac{k-b'+1}{2} = k - \frac{k-8}{2} = \frac{k}{2}+4,$$

whereas

$$c_{b''} = k - r_{b''} \leq k - \frac{k-b''-3}{2} = k - \frac{k-9}{2} = \frac{k}{2}+4\frac{1}{2}.$$

Thus either $c_{b'} > c_{b''}$, or both are equal to $\frac{k}{2}+4$. In the first case, agent 2 can manipulate in profile $(a, b'', c_{b'})$ by reporting b'. This will move the facility from x (where $d(x,b'')=5$) to $b' \pm 1$, which as at most 4 steps from b''.

In the latter case, consider $b^*=5$ and $b^{**}=4$. Computing the critical point for b^{**},

$$c_{b^{**}} = k - r_{b^{**}} \leq k - \frac{k-b^{**}-3}{2} = k - \frac{k-7}{2} < c_{b''}.$$

Thus either $c_{b^*} < c_{b''}$, in which case agent 2 has a manipulation $(a, b^*, c_{b''}) \to b''$, or $c_{b^**} < c_{b^*} = b_{b''}$, in which case agent 2 has a manipulation $(a, b^{**}, c_{b^*}) \to b^*$. □

C.3 Appendix for Sub-Section 4.3

apx:cycle_n

Theorem 13. *Let f be an onto and SP mechanism on R_k, where $k \geq 1$, then f is 1-dictatorial.*

Proof. We assume by induction for every $m < n$ (we know it holds for $n \leq 3$). Let f be an SP unanimous rule for $n \geq 4$ agents. We define two mechanisms for $n-1$ agents:

$$g(a_{-1}) = f(a_1 = a_2, a_{-1}) \quad ; \quad h(a_{-3}) = f(a_3 = a_4, a_{-3}).$$

Now, similarly to the proof of Lemma 10, both g, h are unanimous and SP and therefore both are 1-dictator mechanisms. If we have that some $j \neq 2$ is the dictator of g we are done (since then j is a 1-dictator of f), and similarly for any $j' \neq 4$ in h.

Assume, toward a contradiction that agents 2 and 4 are the 1-dictators of g, h, respectively. Then take any profile where $a_1 = a_2$, $a_3 = a_4$ and $d(a_2, a_4) > 2$ (this is always possible for $k > 4$). We then have that $x = f(\mathbf{a})$ holds both $d(x, a_2) \leq 1$ and $d(x, a_4) \leq 1$, i.e.,$d(a_2, a_4) \leq 2$ in contradiction to the way we defined the profile. □

Corollary 14. *Every SP mechanism on R_k for $k \geq 22$ has an approximation ratio of at least $\frac{n}{2}$.*

Proof. If the mechanism is not unanimous, it has an infinite approximation ratio. Otherwise it is a 1-dictator, w.l.o.g. agent n is the 1-dictator. Let $a_1 = a_2 = \ldots = a_{n-1} = k$, and $a_n = \lfloor\frac{k}{2}\rfloor$. Clearly, the optimal location is opt $= a_1$, and the optimal total distance from all agents is $\lfloor\frac{k}{2}\rfloor$. However, $f(\mathbf{a}) = \lfloor\frac{k}{2}\rfloor \pm 1$, and the total distance from the agents is at least $(n-1)\left(\lfloor\frac{k}{2}\rfloor - 1\right)$ (in fact $\min\{(n-1)\lfloor\frac{k}{2}\rfloor, n\left(\lfloor\frac{k}{2}\rfloor - 1\right)\}$), thus the approximation ratio for $n \geq 3, k \geq 24$ is

$$\frac{SC(f(\mathbf{a}))}{SC(\text{opt})} \geq (n-1)\frac{\lfloor\frac{k}{2}\rfloor - 1}{\lfloor\frac{k}{2}\rfloor} \geq \frac{2n}{3}\frac{3}{4} = \frac{n}{2}.$$

□

C.4 Appendix for Sub-Section 4.4

:cycle_small

Proposition 15. *There is an anonymous SP mechanism for three agents on R_k, for all $k \leq 12$.*

Proof for $k \leq 7$. We define a "median-like" mechanism as follows: let $(a_3, a_1]$ be the longest clockwise arc between agents, then $f(\mathbf{a}) = a_2$. Break ties clockwise, if needed. An agent (say a_1) can try to manipulate. The current dictator (say a_2) must be the one closer a_1, as otherwise $|(a_1, a_2]| > |(a_3, a_1]|$. Thus changing the identity of the dictator from 2 to 3 cannot benefit agent 1. The only way to gain is to become the dictator, by moving away from a_2, making the arc $[a_3, a_1']$ smaller. For agent 1 to become the dictator, the size of $(a_2, a_3]$ must be at least 3, otherwise there is a longer arc (as the sum is $k = 7$). We have that $|(a_3, a_1]| \geq |(a_2, a_3]| \geq 3$, since agent 2 is the dictator for $\mathbf{a}$. As a result, $|(a_1, a_2]| \leq 7 - 3 - 3 = 1$, i.e. either $a_1 = a_2 = f(\mathbf{a})$ (in which case clearly there is no manipulation), or $a_1 = a_2 - 1$. In the latter case, we will have that $f(\mathbf{a}') = a_1' \neq a_1$, so it is still not an improvement for agent 1.

It is easy to verify that the mechanism also works for smaller cycles. □

D Appendix for Section 5

apx:cube

e_cube_props

Lemma 24. *Let f be a mechanism on L_k.*

1. *f is monotone iff f is cube-monotone.*
2. *f is m-SI iff f is m-IIA.*
3. *f is DI iff f is IDA.*
4. *f is Pareto iff f is Cube-Pareto.*

Proof. All equalities follow directly from distance preserving. □

Theorem 17. *Let $k \geq 13$. An onto mechanism on the cycle R_{2k} is SP if and only if it is 1-dictatorial, CMON, and IDA.*

Proof. For the first direction, we must prove that every onto SP mechanism must be IDA and CMON (in addition to being 1-dictatorial). Suppose that some agent violates CMON on coordinate i. This is either by crossing the location of the facility (i.e. $x \in [a_j, a_j']$), or the antipodal point (i.e. $x + k \in [a_j, a_j']$). In the first case, this is clearly a manipulation, as shown in Lemma 3. In the latter case, assume w.l.o.g. that $x = 0$ and j moved clockwise. By Lemma 6, the agent moved along the longer arc $[a_j, x)$, thus $|[x, a_j)| \leq k$. By violation of CMON, the facility also moved clockwise, thus getting closer to a_j, which is a manipulation.

Now suppose IDA is violated by agent j (moving w.l.o.g. clockwise). This means that $[a_j, a_j']$ does not contain neither x nor $k + x$. If j is the dictator, then this is clearly a violation (again, as in Theorem 2). Otherwise, the facility can only move one or two steps. If the facility moves clockwise, then $a_j \to a_j'$ is a manipulation. Otherwise, $a_j' \to a_j$ is a manipulation.

In the other direction, we show that if f is 1-dictatorial (where agent 1 is the dictator), IDA and CMON, then it must be SP. Indeed, suppose that agent j moves from a_j to a_j', thereby moving the facility from x to x', where w.l.o.g. $x = 0$. If $j = 1$, then the only way to gain is by moving one step, bringing the facility to a_1. However this would contradict IDA. Therefore assume that j is not the dictator. By Lemma 6, j must move along the longer arc $[a_j, x]$. Consider first $a_1 = x = 0$. Note that the facility can only move to $x' = 2k - 1$ (if $x'(k) \neq x(k)$) or $x' = 1$ (if $x(1) \neq x'(1)$).

(I) If j is not crossing neither $x = 0 = 0^k$, nor $k = 1^k$. Then by IDA coordinates 1 and k of the facility cannot change. Since changing any other coordinate puts x' at distance at least 2 from the dictator, the facility cannot move.

(II) If $a_j = x$, then j cannot gain.

(III) $k \in [a_j, a'_j]$ (w.l.o.g. j is moving clockwise, from $a_j \in [1, k]$ to $a'_j \in [k, 2k-1]$). We part the movement to $a_j \to k$, and $k \to a'_j$, and denote $f(a_{-j}, k) = y$. In the first part, only coordinates between 2 and k of a_j change (from 0 to 1). Then by IDA $y(1) = x(1) = 0$, and the facility can only move to $y = 2k-1$. However this means that $d(y, a_j) > d(x, a_j)$, i.e. that j does not gain.

Now, if $y = 2k-1$, then by IDA $x' = 2k-1$ as well. Therefore suppose that $y = x = 0$. Since only coordinates $< k$ change between k and a'_j, $x'(k) = y(k)$. On the other hand, $a'_j(1) = 0 < k(1)$, thus by CMON $x'(1) \leq x(1) = 0$ as well. Therefore, $x' = x$.

It is left to handle the case where $a_1 \neq x$. Since agent 1 is a 1-dictator $d(a_1, x) = 1$, and $d(a_1, x') \leq 1$. The only difference is that in case (III) it is possible that $y = 2k-2$ (if $a_1 = 2k-1$). However this is still not a manipulation, as $d(y, a_j) \geq d(x, a_j)$ (rather than strictly larger). □

E Appendix for Section 6

x:discussion

The most prominent concept class is perhaps the class of *linear classifiers* in $\mathbb{R}^d$. A linear classifier is composed of a unit vector $\mathbf{w} \in \mathbb{R}^d$ and a scalar u. For it classifies every data point $\mathbf{x} \in \mathbb{R}^d$, to $\{+, -\}$, according to $sign(\langle \mathbf{w}, \mathbf{x} \rangle - u)$. A linear classifier in $\mathbb{R}^1$ is just a scalar u and a direction $w \in \{-, +\}$.

Suppose that the dataset contains k samples in generic state on the real line $\mathbb{R}^1$. There are exactly $2k$ ways to classify the points, as all negative labels must be on one side of the classifier and likewise for the positive labels. Requiring *realizability* simply means that the opinion of each agent on the correct classification can be described by one of these classifiers.

The cycle R_k contains the $2k$ vectors $0^{k_1}1^{k_2}$ and $1^{k_1}0^{k_2}$, where $k_1 + k_2 = k$. Note that these are exactly all the possibilities to classify the dataset with a 1-dimensional linear classifier. We can map the classification problem to a facility location by mapping each data point to a particular dimension of the cube C_k. The opinion of each agent can be naturally mapped to a vertex in C_k. Moreover, due to realizability, this vertex lies on R_k. We get that any SP mechanism for classification is in fact an SP facility location mechanism on the cycle R_{2k} (the facility is the selected classifier). The following corollary then follows.

Corollary 25. *Every SP classification mechanism for linear classifiers in $\mathbb{R}^d$ (for any $d \geq 1$) has an approximation ratio of $\Omega(n)$.*

For a more detailed discussion on SP linear classifiers (including a simple reduction from $\mathbb{R}^d$ to $\mathbb{R}^1$), see Meir et al. [14].