Exact and Efficient Construction of Minkowski Sums of Convex Polyhedra with Applications

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Outline

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Introduction

- P and Q are two convex polyhedra (polytopes) in $\mathbb{R}^d$.

$$M = P \oplus Q = \{p + q \,|\, p \in P, q \in Q\} \quad \text{Minkowski sum}$$

An algorithm that Computes the Minkowski sum:

- Compute the Minkowski sum of *a set of polytopes*.
- Complete
  - Exact results are guaranteed.
  - All degenerate cases are handled.
- Use a dual representation — *Cubical Gaussian Map*.
- Implemented on top of CGAL.
  - Arrangement and Polyhedral Surface packages.

Introduction

- P and Q are two convex polyhedra in $\mathbb{R}^d$.
  - P translated by a vector t is denoted by P^t.

$$\delta(P,Q) = \min\{\|t\| \mid P^t \cap Q \neq \emptyset, t \in \mathbb{R}^d\}$$ minimum distance

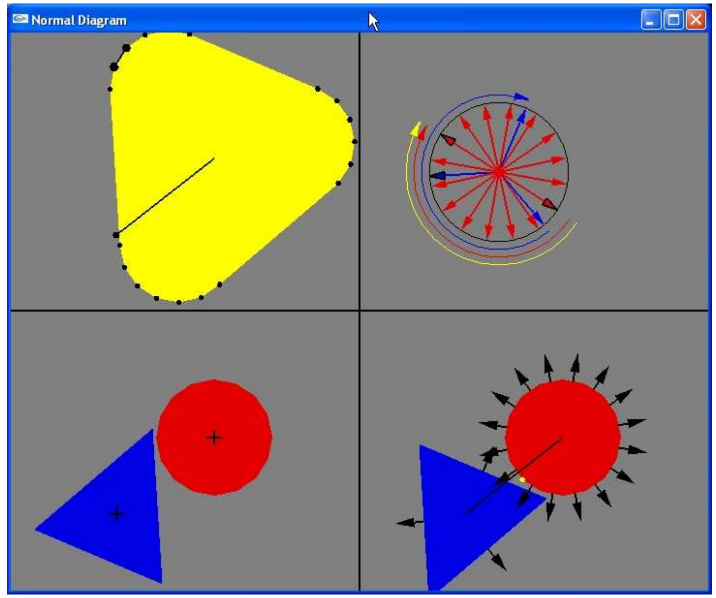
$$\pi(P,Q) = \min\{\|t\| \mid P^t \cap Q = \emptyset, t \in \mathbb{R}^d\}$$ penetration depth

$$\pi_d(P,Q) = \min\{a \mid P^{\vec{d}a} \cap Q = \emptyset\}$$ directional penetration depth

- The minimum separation distance between P and Q is the same as the minimum distance between the origin and the boundary of the Minkowski sum of P and $-Q$.

Warmup —2D Collision Detection

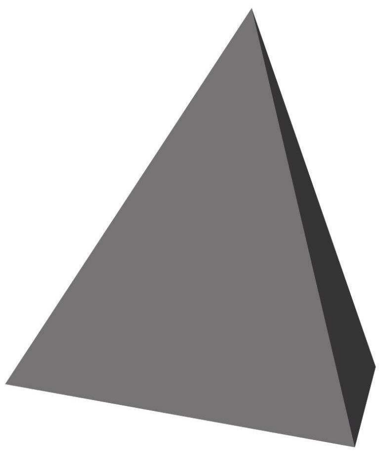
An interactive program that simulates and demonstrates 2D collision detection.



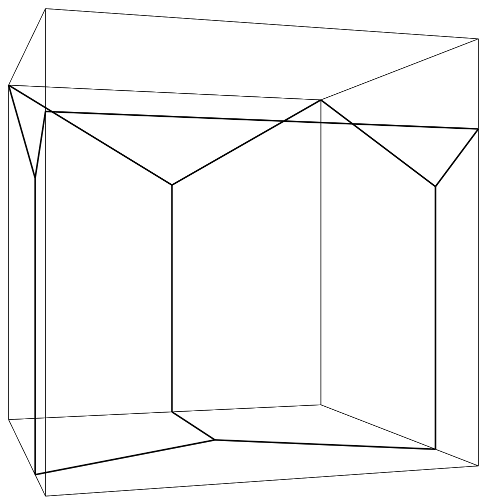
The Cubical Gaussian Map

The *Cubical Gaussian Map* (CGM) C of a polytope P in $\mathbb{R}^3$ is a set-valued function from P to the six faces of the unit cube whose edges are parallel to the major axes and are of length two.

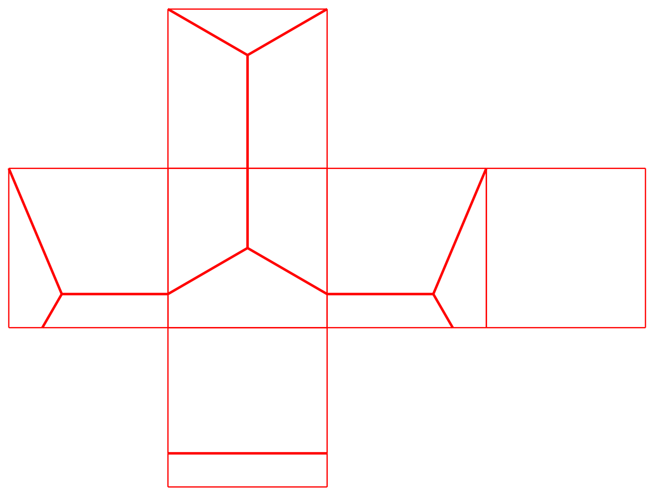
A Tetrahedron



The primal



The CGM



The CGM unfolded

Exact Minkowski Sums of Convex Polyhedra

The overlay of the Gaussian maps of two polytopes P and Q identifies all the pairs of features of P and Q respectively that have common supporting planes, as they occupy the same space on the unit sphere, thus, identifying all the pairwise features that contribute to the boundary of the Minkowski sum of P and Q.

- Each planar map in a CGM is a convex subdivision.
- The overlay can be computed in $O(n + m)$ time.
- We compute it in $O(k \log(m + n))$ time.
    - m, n, k — number of facets in P, Q, $P \oplus Q$.

Exact Minkowski Sums of Convex Polyhedra

Input: $P_0, P_1, \ldots, P_r$ convex polyhedra in CGM representation.

Output: $P = P_0 \oplus P_1 \oplus \ldots \oplus P_r$ in CGM representation.

1: $P \leftarrow P_0$

2: **for** $i = 1$ to r **do**

2.1: Insert all edges of P_i into P.

3: **for** $i = 0$ to r **do**

3.1: Add the appropriate vertex of P_i to the Minkowski-sum vertex stored in each planar-map face of P.

4: Compute additional information stored in boundary halfedges and boundary vertices.

5: Compute equations of the planes containing the Minkowski-sum facets for all vertices.

3D Collision Detection

$$P^u \cap Q^w \neq \emptyset \Leftrightarrow v - u \in M = P \oplus (-Q), \tag{1}$$

$$\delta(P^u, Q^w) = \min\{\|t\| \,|\, (v - u + t) \in M, t \in \mathbb{R}^3\}, \tag{2}$$

$$\pi_d(P^u, Q^w) = \min\{\alpha \,|\, (v - u + \vec{d}\alpha) \notin M\}. \tag{3}$$

Experimental Results

- The number of features of the six planar maps of the CGM of the squashed dioctagonal pyramid polytope

Planar map	**V**	**HE**	**F**
0, ($x = -1$)	12	32	6
1, ($y = -1$)	36	104	18
2, ($z = -1$)	12	32	6
3, ($x = 1$)	12	32	6
4, ($y = 1$)	21	72	17
5, ($z = 1$)	12	32	6
Total	105	304	59

Experimental Results

- Complexity of the primal and dual representations

Object Name	Primal			Dual		
	V	E	F	V	HE	F
Tetrahedron	4	6	4	38	94	21
Octahedron	6	12	6	24	48	12
Icosahedron	12	30	20	72	192	36
Dioctagonal Pyramid	17	32	17	105	304	59
Pentagonal Hexecontahedron	92	150	60	196	684	158
Truncated Icosidodecahedron	120	180	62	230	840	202
Geodesic Sphere level 4	252	750	500	708	2124	366

Experimental Results

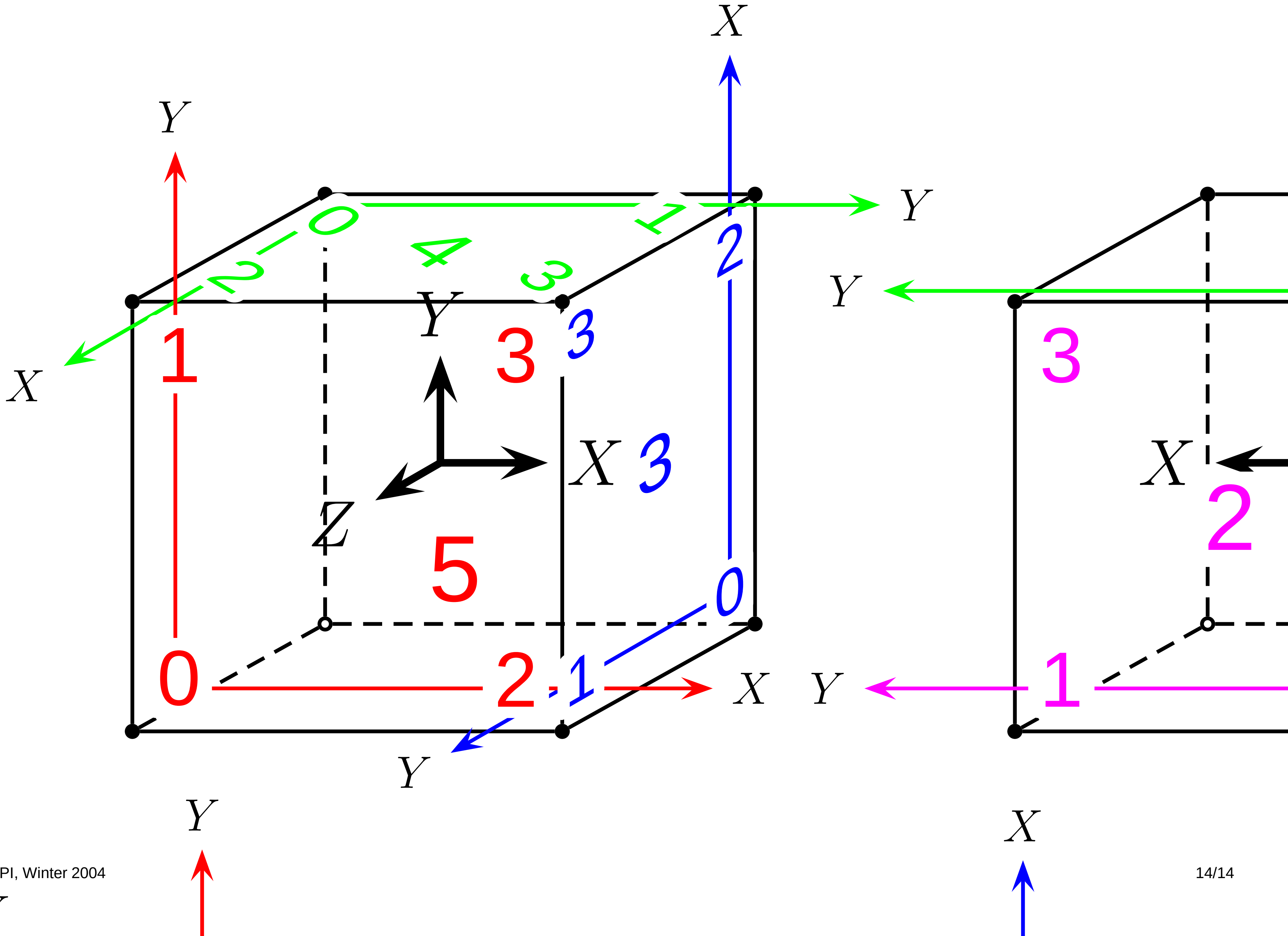
- Time consumption of Minkowski-sum computation

Object 1	Object 2	Minkowski Sum						CH	CGMO
		Primal			Dual				
		V	E	F	V	HE	F		
Icosahedron	Icosahedron	12	30	20	72	192	36	0.13	0.03
Dioctagonal Pyramid	Orthogonal Dioctagonal Pyramid	162	352	192	344	1136	236	0.53	0.28
Pentagonal Hexeconta-hedron	Truncated Icosidodec-ahedron	527	964	439	719	2744	665	16.76	1.21
Geodesic Sphere level 4	Rotated Geodesic Sphere level 4	814	1952	1146	1530	5060	1012	134.35	2.53

Future Work

- Exact rotation
- Compute only relevant portions of the Minkowski sum
  - Walk simultaneously on the two respective CGMs
- Adapt to rotation
  - Rotate the trajectory of the walk instead of the CGM

Data Structure



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