Minkowski Sums of Convex Polyhedra and Applications

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Minkowski Sums and Proximity Queries

- P and Q are two convex polyhedra (polytopes) in $\mathbb{R}^d$

$$M = P \oplus Q = \{p + q \mid p \in P, q \in Q\} \qquad \text{Minkowski sum}$$

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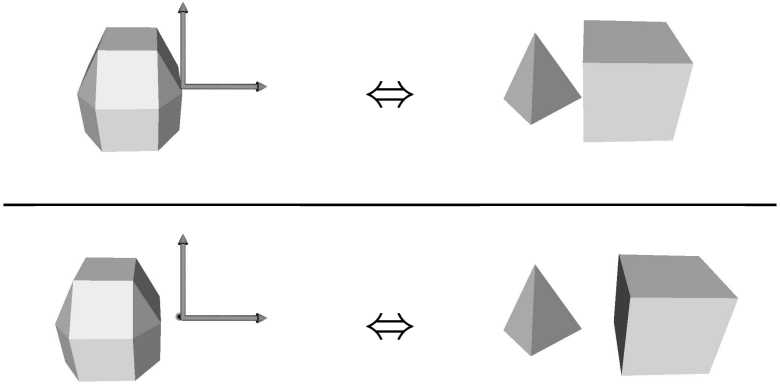
$$P \cap Q \neq \emptyset \Leftrightarrow \text{Origin} \in M = P \oplus (-Q) \qquad \text{collision detection}$$

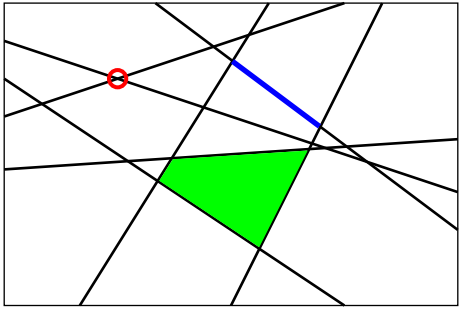


Outline

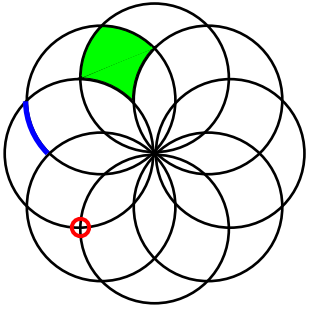
- Arrangements
- Minkowski Sums and Proximity Queries
- Gaussian Maps
- Computing The Exact Minkowski Sums
- Experiment Results
- About the Exact Complexity of Minkowski Sums
- Demo + Movie

Arrangements in $\mathbb{R}^2$

Given a collection Γ of planar curves, the arrangement $\mathcal{A}(\Gamma)$ is the partition of the plane into vertices, edges and faces induced by the curves of Γ.



An arrangement of lines



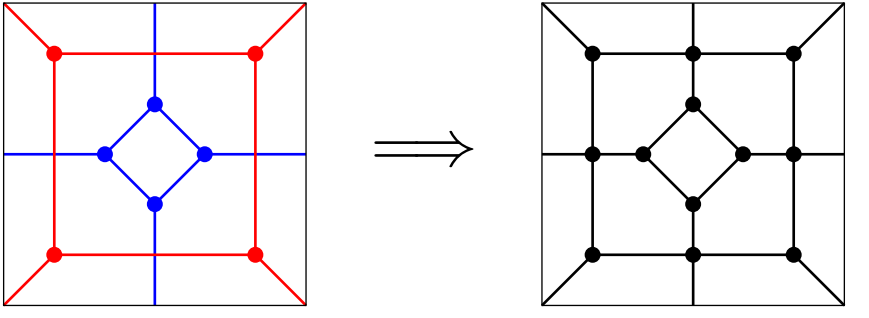
An arrangement of circles

The `Arrangement_2` Package

- Part of the basic library of CGAL
- Used to construct, maintain, modify, traverse, query and present arrangements
- Offers a class template `Arrangement_2` that represents an arrangement
- Supports free functions that operate on `Arrangement_2` objects (e.g., `overlay(arr1, arr2, res)`)
- Efficient
- Robust — all (degenerate) inputs are handled correctly
- Easy to use, extend, and adapt

Arrangement Overlay

The overlay of two arrangements $\mathcal{A}_1$ and $\mathcal{A}_2$ is an arrangement $\mathcal{A}$ such that there is a face f in $\mathcal{A}$ if and only if there are faces f_1 and f_2 in $\mathcal{A}_1$ and $\mathcal{A}_2$ respectively such that f is a maximal connected subset of $f_1 \cap f_2$.



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  - P translated by a vector t is denoted by P^t

$$M = P \oplus Q = \{p + q \mid p \in P, q \in Q\}$$ Minkowski sum

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$$\delta(P, Q) = \min\{\|t\| \mid P^t \cap Q \neq \emptyset, t \in \mathbb{R}^d\}$$ minimum distance

$$\delta(P^u, Q^w) = \min\{\|t\| \mid (w - u + t) \in M, t \in \mathbb{R}^3\}$$

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$$\pi(P, Q) = \inf\{\|t\| \mid P^t \cap Q = \emptyset, t \in \mathbb{R}^d\}$$ penetration depth

$$\pi_d(P, Q) = \inf\{a \mid P^{\vec{da}} \cap Q = \emptyset\}$$ directional penetration depth

$$\pi_d(P^u, Q^w) = \inf\{\alpha \mid (w - u + \vec{d}\alpha) \notin M\}$$

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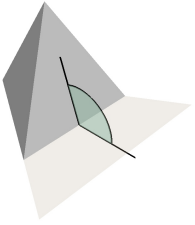
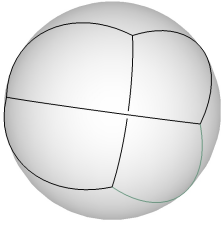
Spherical Gaussian Map

The *Gaussian Map* G of a compact convex polyhedron P in Euclidean three-dimensional space $\mathbb{R}^3$ is a set-valued function from P to the unit sphere $\mathbb{S}^2$, which assigns to each point p the set of outward unit normals to support planes to P at p.

A Tetrahedron

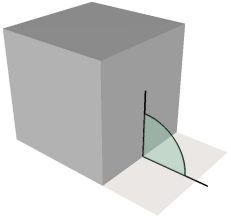


Primal

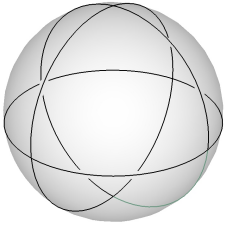


Gaussian map

A Cube



Primal

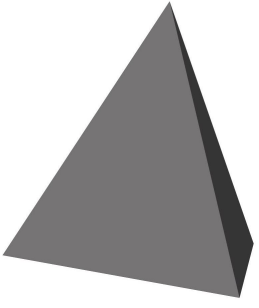


Gaussian map

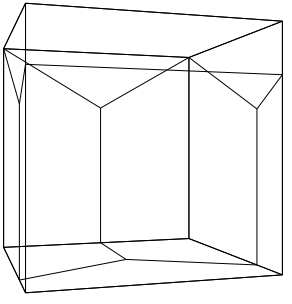
Cubical Gaussian Map

Given a polytope P in $\mathbb{R}^3$ The *Cubical Gaussian Map* (CGM) $\mathcal{C}(P)$ is the set-valued function from P to the 6 faces of the unit cube* whose edges are parallel to the major axes and are of length two.

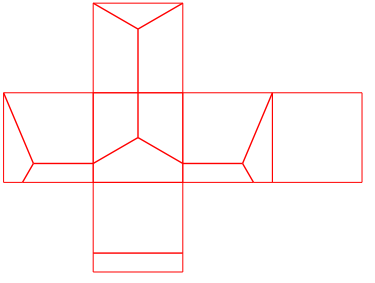
A Tetrahedron



The primal



The CGM



The CGM unfolded

(*) — The unit cube is the unit sphere in L^∞

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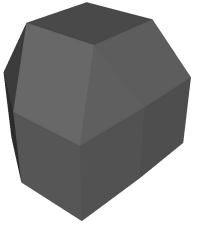
Spherical Gaussian Map of Minkowski Sums

The overlay of the Gaussian maps of two polytopes P and Q in $\mathbb{R}^3$ identifies all the pairs of features of P and Q respectively that have common supporting planes, as they occupy the same space on the unit sphere, thus, identifying all the pairwise features that contribute to the boundary of the Minkowski sum of P and Q.

The Minkowski sum of a tetrahedron and a cube



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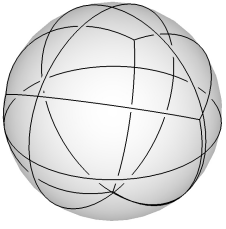
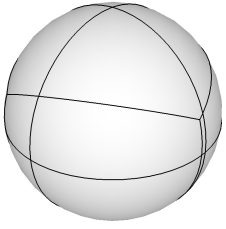


The transparent SGM

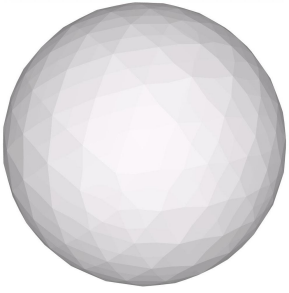


The opaque SGM

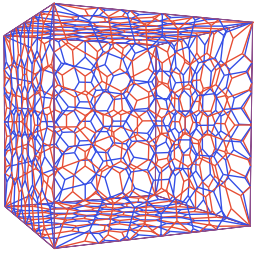
The Cubical Gaussian Map of Minkowski Sums

Amounts to computing the overlay of six arrangement pairs.

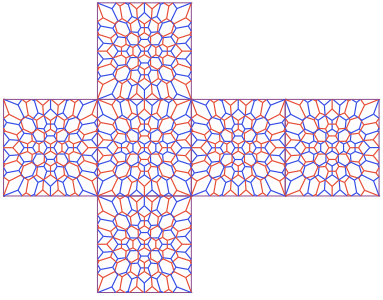
The Minkowski sum of two geodesic spheres level 4 slightly rotated with respect to each other.



The primal

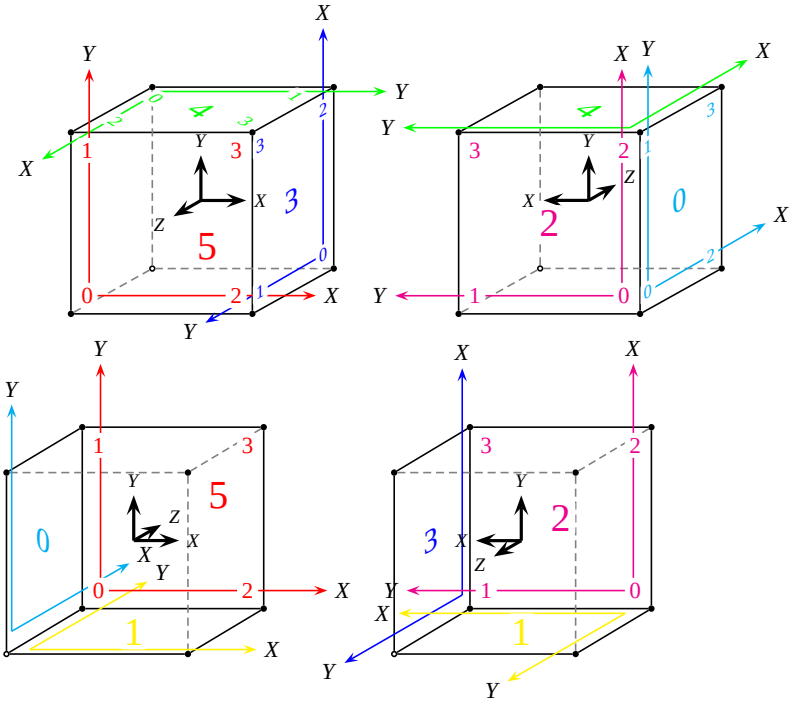


The CGM



The CGM unfolded

The Data Structure



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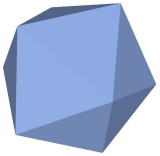
Complexities of Polytopes in $\mathbb{R}^3$



Icos

DP

PH

TI

GS4

Object Name	**Primal**			**Dual**		
	V	**E**	**F**	**V**	**HE**	**F**
Icosahedron (Icos)	12	30	20	72	192	36
Dioctagonal Pyramid (DP)	17	32	17	105	304	59
Pentagonal Hexecontahedron (PH)	92	150	60	196	684	158
Truncated Icosidodecahedron (TI)	120	180	62	230	840	202
Geodesic Sphere level 4 (GS4)	252	750	500	708	2124	366

Minkowski Sums of Polytopes in $\mathbb{R}^3$

Obj. 1	Obj. 2	Minkowski Sum						CGM	NGM	LP	CH	$\frac{F_1 F_2}{F}$
		Primal			Dual							
		V	E	F	V	HE	F					
Icos	Icos	12	30	20	72	192	36	**0.01**	0.36	0.04	0.10	20
DP	ODP	131	261	132	242	832	186	**0.02**	1.08	0.35	0.31	2.2
PH	TI	248	586	340	514	1670	333	**0.05**	2.94	1.55	3.85	10.9
GS4	RGS4	1048	2568	1531	1906	6288	1250	**0.31**	14.33	5.80	107.35	163.3

Time consumption (in seconds) of various Minkowski-sum computations

CGM — the Cubical Gaussian Map based method
NGM — the Nef polyhedra based method [Hachenberger, Kettner]
LP — Linear Programming based algorithm [Fukuda, Weibel]
CH — the Convex Hull method
$\frac{F_1 F_2}{F}$ — the ratio between the product of the number of input facets and the number of output facets

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Minkowski Sum Exact Complexity

Theorem 1. *The exact maximum number of facets in the Minkowski sum $M = P \oplus Q$ of two polytopes P and Q with m and n facets respectively is*
$f(m, n) = 4mn - 9m - 9n + 26.$

Upper Bound — The complexity cannot exceed $f(m, n)$.

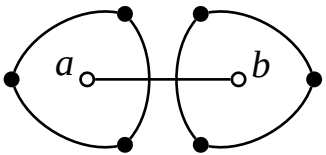
Lower Bound — For every m and n there exist P and Q, such that their Minkowski-sum complexity is $f(m, n)$.

Gaussian Map Properties

Proposition 2. *Let G_1 and G_2 be two Gaussian maps, and let G be their overlay. Let f_1, f_2, and f denote the number of faces of G_1, G_2, and G respectively. Then, the number of faces f of G cannot exceed $f_1 \cdot f_2$.*

A face of G_1 (resp. G_2) is convex in the following sense:

$$a \in G_i \text{ and } b \in G_i \Longrightarrow \widehat{ab} \in G_i$$



Contradiction!

$\widehat{ab}$ is a unique geodesic segment.

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