Minkowski Sums of Convex Polyhedra and Applications

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Minkowski Sums and Proximity Queries

- $P$ and $Q$ are two convex polyhedra (polytopes) in $\mathbb{R}^d$

\[ M = P \oplus Q = \{ p + q \mid p \in P, q \in Q \} \]

Minkowski sum
Minkowski Sums and Proximity Queries

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Minkowski sum

\( P \cap Q \neq \emptyset \iff \text{Origin} \in M = P \oplus (-Q) \)

collision detection

\[
\text{Origin} \in M = P \oplus (-Q)
\]
Outline

- Arrangements
- Minkowski Sums and Proximity Queries
- Gaussian Maps
- Computing The Exact Minkowski Sums
- Experiment Results
- About the Exact Complexity of Minkowski Sums
- Demo + Movie
Arrangements in $\mathbb{R}^2$

Given a collection $\Gamma$ of planar curves, the arrangement $\mathcal{A}(\Gamma)$ is the partition of the plane into vertices, edges and faces induced by the curves of $\Gamma$.

An arrangement of lines  An arrangement of circles
The Arrangement_2 Package

- Part of the basic library of CGAL
- Used to construct, maintain, modify, traverse, query and present arrangements
- Offers a class template `Arrangement_2` that represents an arrangement
- Supports free functions that operate on `Arrangement_2` objects (e.g., `overlay(arr1, arr2, res)`)
- Efficient
- Robust — all (degenerate) inputs are handled correctly
- Easy to use, extend, and adapt

Minkowski Sums
Arrangement Overlay

The overlay of two arrangements $\mathcal{A}_1$ and $\mathcal{A}_2$ is an arrangement $\mathcal{A}$ such that there is a face $f$ in $\mathcal{A}$ if and only if there are faces $f_1$ and $f_2$ in $\mathcal{A}_1$ and $\mathcal{A}_2$ respectively such that $f$ is a maximal connected subset of $f_1 \cap f_2$. 

\[\text{Minkowski Sums}\]
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Minkowski Sums and Proximity Queries

- $P$ and $Q$ are two convex polyhedra (polytopes) in $\mathbb{R}^d$
  - $P$ translated by a vector $t$ is denoted by $P^t$

$$M = P \oplus Q = \{ p + q \mid p \in P, q \in Q \}$$

Minkowski sum
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$P^u \cap Q^w \neq \emptyset \iff w - u \in M = P \oplus (-Q)$

Minkowski sum

collision detection
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\[ M = P \oplus Q = \{ p + q \mid p \in P, q \in Q \} \]

\( P^u \cap Q^w \neq \emptyset \iff w - u \in M = P \oplus (-Q) \)

\[ \delta(P, Q) = \min\{\|t\| \mid P^t \cap Q \neq \emptyset, t \in \mathbb{R}^d \} \]

\[ \delta(P^u, Q^w) = \min\{\|t\| \mid (w - u + t) \in M, t \in \mathbb{R}^3 \} \]

Minkowski sum

collision detection

minimum distance
Minkowski Sums and Proximity Queries

- $P$ and $Q$ are two convex polyhedra (polytopes) in $\mathbb{R}^d$
  - $P$ translated by a vector $t$ is denoted by $P^t$

  \[
  M = P \oplus Q = \{ p + q \mid p \in P, q \in Q \} \quad \text{Minkowski sum}
  \]

  \[
  P^u \cap Q^w \neq \emptyset \iff w - u \in M = P \oplus (-Q) \quad \text{collision detection}
  \]

  \[
  \delta(P, Q) = \min \{ \|t\| \mid P^t \cap Q \neq \emptyset, t \in \mathbb{R}^d \} \quad \text{minimum distance}
  \]

  \[
  \delta(P^u, Q^w) = \min \{ \|t\| \mid (w - u + t) \in M, t \in \mathbb{R}^3 \} \quad \text{penetration depth}
  \]

  \[
  \pi(P, Q) = \inf \{ \|t\| \mid P^t \cap Q = \emptyset, t \in \mathbb{R}^d \} \quad \text{penetration depth}
  \]

  \[
  \pi_d(P, Q) = \inf \{ a \mid P^{d\alpha} \cap Q = \emptyset \} \quad \text{directional penetration depth}
  \]

  \[
  \pi_d(P^u, Q^w) = \inf \{ \alpha \mid (w - u + d\alpha) \notin M \}
  \]
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**Spherical Gaussian Map**

The *Gaussian Map* $G$ of a compact convex polyhedron $P$ in Euclidean three-dimensional space $\mathbb{R}^3$ is a set-valued function from $P$ to the unit sphere $S^2$, which assigns to each point $p$ the set of outward unit normals to support planes to $P$ at $p$. 

![Diagram of a Tetrahedron and a Cube: Primal and Gaussian map versions]
Cubical Gaussian Map

Given a polytope $P$ in $\mathbb{R}^3$ The Cubical Gaussian Map (CGM) $\mathcal{C}(P)$ is the set-valued function from $P$ to the 6 faces of the unit cube* whose edges are parallel to the major axes and are of length two.

A Tetrahedron

The primal  
The CGM  
The CGM unfolded

(*) — The unit cube is the unit sphere in $L^\infty$
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Spherical Gaussian Map of Minkowski Sums

The overlay of the Gaussian maps of two polytopes $P$ and $Q$ in $\mathbb{R}^3$ identifies all the pairs of features of $P$ and $Q$ respectively that have common supporting planes, as they occupy the same space on the unit sphere, thus, identifying all the pairwise features that contribute to the boundary of the Minkowski sum of $P$ and $Q$.

The Minkowski sum of a tetrahedron and a cube
The Cubical Gaussian Map of Minkowski Sums

Amounts to computing the overlay of six arrangement pairs.

The Minkowski sum of two geodesic spheres level 4 slightly rotated with respect to each other.
The Data Structure

Minkowski Sums
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Complexities of Polytopes in $\mathbb{R}^3$

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<th>Object Name</th>
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Minkowski Sums
Minkowski Sums of Polytopes in $\mathbb{R}^3$

<table>
<thead>
<tr>
<th>Obj. 1</th>
<th>Obj. 2</th>
<th>Minkowski Sum</th>
<th>CGM</th>
<th>NGM</th>
<th>LP</th>
<th>CH</th>
<th>$\frac{F_1 F_2}{F}$</th>
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</table>

Time consumption (in seconds) of various Minkowski-sum computations

**CGM** — the Cubical Gaussian Map based method

**NGM** — the Nef polyhedra based method [Hachenberger, Kettner]

**LP** — Linear Programming based algorithm [Fukuda, Weibel]

**CH** — the Convex Hull method

$\frac{F_1 F_2}{F}$ — the ratio between the product of the number of input facets and the number of output facets
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Minkowski Sum Exact Complexity

**Theorem 1.** The exact maximum number of facets in the Minkowski sum $M = P \oplus Q$ of two polytopes $P$ and $Q$ with $m$ and $n$ facets respectively is

$$f(m, n) = 4mn - 9m - 9n + 26.$$  

**Upper Bound** — The complexity cannot exceed $f(m, n)$.

**Lower Bound** — For every $m$ and $n$ there exist $P$ and $Q$, such that their Minkowski-sum complexity is $f(m, n)$. 
Gaussian Map Properties

**Proposition 2.** Let $G_1$ and $G_2$ be two Gaussian maps, and let $G$ be their overlay. Let $f_1$, $f_2$, and $f$ denote the number of faces of $G_1$, $G_2$, and $G$ respectively. Then, the number of faces $f$ of $G$ cannot exceed $f_1 \cdot f_2$.

A face of $G_1$ (resp. $G_2$) is convex in the following sense:

$$a \in G_i \text{ and } b \in G_i \implies \widehat{ab} \in G_i$$

$\widehat{ab}$ is a unique geodesic segment.

Contradiction!
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Demo + Movie
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