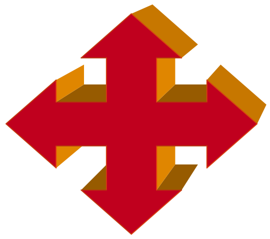
2D Geometric Transformations

(Chapter 5 in FVD)



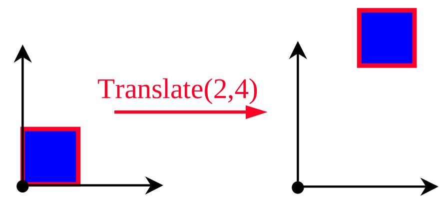
2D Geometric Transformations

- **Question**: How do we represent a geometric object in the plane?
- **Answer**: For now, assume that objects consist of points and lines. A point is represented by its Cartesian coordinates: (x,y).

- **Question**: How do we transform a geometric object in the plane?
- **Answer**: Let (A,B) be a straight line segment and T a general 2D transformation: T transforms (A,B) into another straight line segment (A',B'), where $A'=TA$ and $B'=TB$.

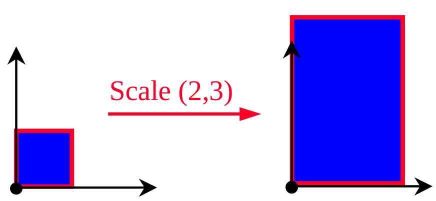
Translation

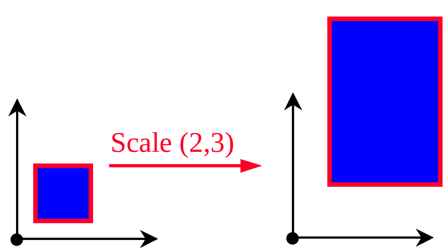
- Translate (a,b): $(x,y) \rightarrow (x+a,y+b)$



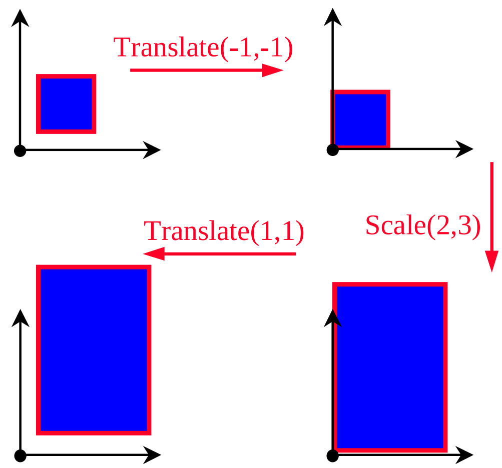
Scale

- Scale (a,b): $(x,y) \rightarrow (ax,by)$

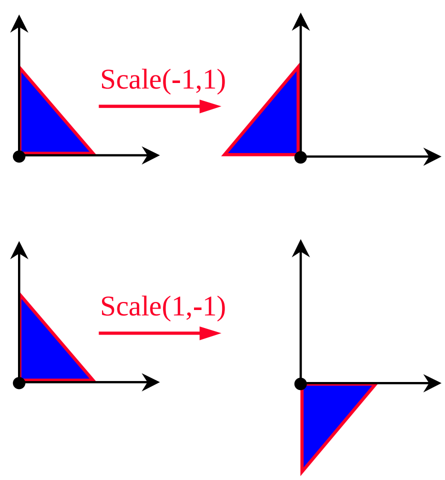




- How can we scale an object without moving its origin (lower left corner)?



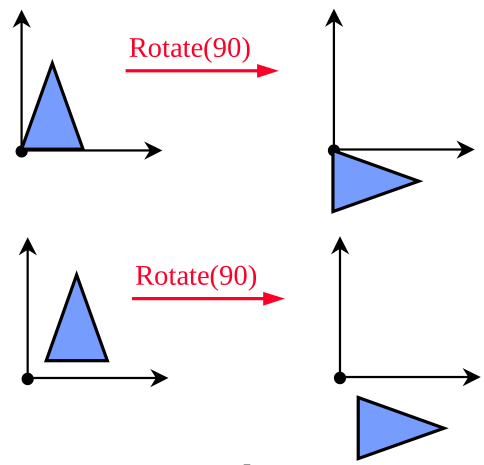
Reflection



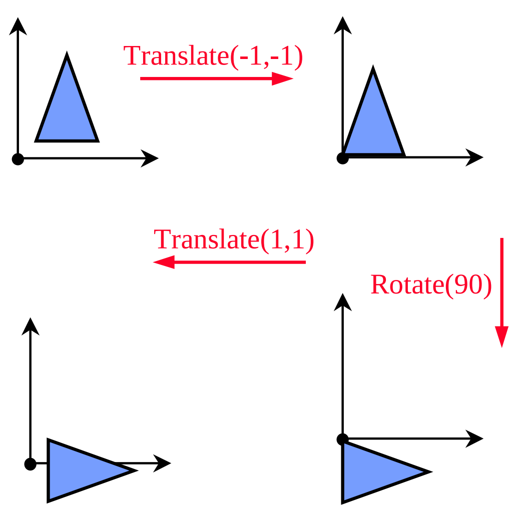
Rotation

- Rotate(θ):

$(x,y) \rightarrow (x\cos(\theta)+y\sin(\theta),\ -x\sin(\theta)+y\cos(\theta))$

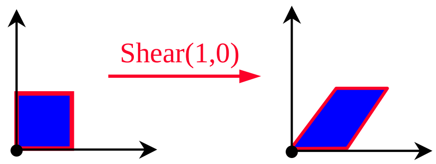


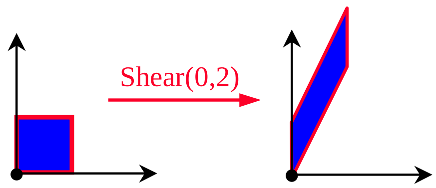
- How can we rotate an object without moving its origin (lower left corner)?



Shear

- Shear (a,b): (x,y) → (x+ay,y+bx)





Composition of Transformations

- Rigid transformation:
  - Translation + Rotation (distance preserving).
- Similarity transformation:
  - Translation + Rotation + uniform Scale (angle preserving).
- Affine transformation:
  - Translation + Rotation + Scale + Shear (parallelism preserving).
- All above transformations are groups where Rigid ⊂ Similarity ⊂ Affine.

Matrix Notation

- Let's treat a point (x,y) as a 2x1 matrix (a column vector):

$$\begin{bmatrix} x \\ y \end{bmatrix}$$

- What happens when this vector is multiplied by a 2x2 matrix?

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

2D Transformations

- 2D object is represented by points and lines that join them.
- Transformations can be applied only to the the points defining the lines.
- A point (x,y) is represented by a 2x1 column vector, and we can represent 2D transformations using 2x2 matrices:

$$\begin{bmatrix} x' \end{bmatrix} = \begin{bmatrix} a & b \end{bmatrix}\begin{bmatrix} x \end{bmatrix}$$

Scale

- Scale(a,b): $(x,y) \longrightarrow (ax,by)$

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax \\ by \end{bmatrix}$$

- If *a* or *b* are negative, we get reflection.

Reflection

- Reflection through the *y* axis:

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

- Reflection through the *x* axis:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

- Reflection through $y=x$:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

- Reflection through $y=-x$:

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

Shear, Rotation

- Shear(a,b): (x,y) $\longrightarrow$ (x+ay,y+bx)

$$\begin{bmatrix} 1 & a \\ b & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + ay \\ y + bx \end{bmatrix}$$

- Rotate(θ):

  (x,y) $\longrightarrow$ ($x$cosθ+$y$sinθ , -$x$sinθ + $y$cosθ)

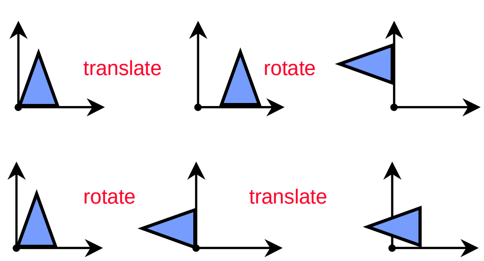
$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\cos\theta + y\sin\theta \\ -x\sin\theta + y\cos\theta \end{bmatrix}$$

Composition of Transformations

- A sequence of transformations can be collapsed into a single matrix:

$$[A][B][C]\begin{bmatrix} x \\ y \end{bmatrix} = [D]\begin{bmatrix} x \\ y \end{bmatrix}$$

- Note: order of transformations is important! (otherwise - commutative groups)



Translation

- Translation(a,b): $\begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} x+a \\ y+b \end{bmatrix}$
- **Problem**: Cannot represent translation using 2x2 matrices.
- **Solution**:

  Homogeneous Coordinates

Homogeneous Coordinates

- Homogeneous Coordinates is a mapping from R^n to R^{n+1}:

$$(x,y) \rightarrow (X,Y,W) = (tx,ty,t)$$

- Note: (tx,ty,t) all correspond to the same non-homogeneous point (x,y). E.g. (2,3,1)≡(6,9,3).
- Inverse mapping:

$$(X,Y,W) \rightarrow \left(\frac{X}{W},\frac{Y}{W}\right)$$

Translation

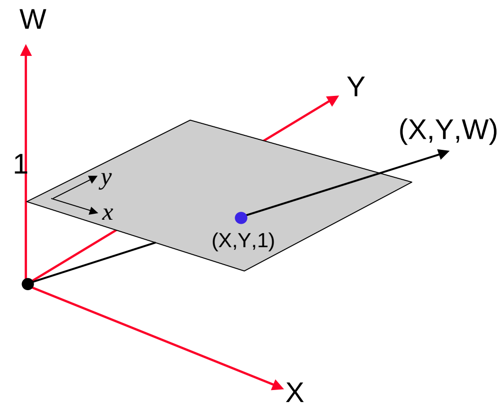
- Translate(a,b):

$$\begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x+a \\ y+b \\ 1 \end{bmatrix}$$

- Affine transformation now have the following form:

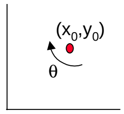
$$\begin{bmatrix} a & b & e \\ c & d & f \\ 0 & 0 & 1 \end{bmatrix}$$

Geometric Interpretation



- A 2D point is mapped to a line (ray) in 3D. The non-homogeneous points are obtained by projecting the rays onto the plane Z=1.

- Example: Rotation about an arbitrary point:

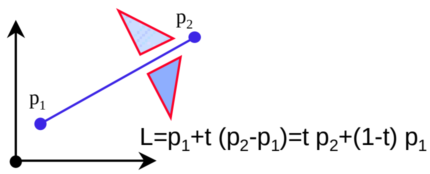


- Actions:
  - Translate the coordinates so that the origin is at (x_0,y_0).
  - Rotate by θ.
  - Translate back.

$$\begin{bmatrix}1 & 0 & x_0\\ 0 & 1 & y_0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & -x_0\\ 0 & 1 & -y_0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ 1\end{bmatrix}=$$

$$=\begin{bmatrix}\cos\theta & -\sin\theta & x_0(1-\cos\theta)+y_0\sin\theta\\ \sin\theta & \cos\theta & y_0(1-\cos\theta)-x_0\sin\theta\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ 1\end{bmatrix}$$

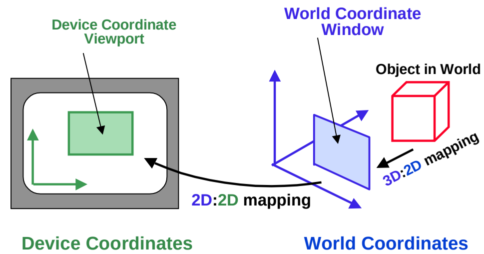
- Another example: Reflection about an Arbitrary Line:



- Actions:
  - Translate the coordinates so that P_1 is at the origin.
  - Rotate so that L aligns with the x-axis.
  - Reflect about the x-axis.
  - Rotate back.
  - Translate back.

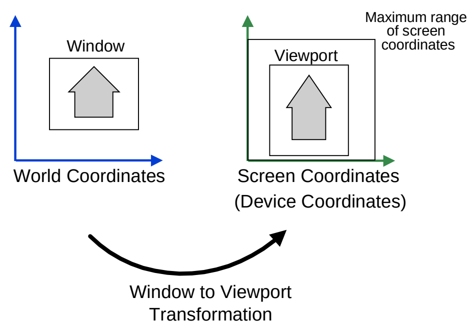
Viewing in 2D

(Chapter 6 in FVD)

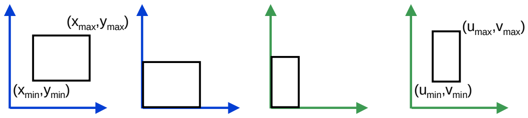


- Objects are given in *world coordinates*.
- The world is viewed through a *world-coordinate window*.
- The WC window is mapped onto a *device coordinate viewport*.

Viewing in 2D (cont.)



Window to Viewport Transformation



$$M_{wv} = T(u_{min},v_{min})\ S\left(\frac{u_{max}-u_{min}}{x_{max}-x_{min}}, \frac{v_{max}-v_{min}}{y_{max}-y_{min}}\right)\ T(-x_{min},-y_{min})$$

$$= \begin{bmatrix} 1 & 0 & u_{min} \\ 0 & 1 & v_{min} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{u_{max}-u_{min}}{x_{max}-x_{min}} & 0 & 0 \\ 0 & \frac{v_{max}-v_{min}}{y_{max}-y_{min}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -x_{min} \\ 0 & 1 & -y_{min} \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{u_{max}-u_{min}}{x_{max}-x_{min}} & 0 & \frac{u_{max}-u_{min}}{x_{max}-x_{min}}(-x_{min}) + u_{min} \\ 0 & \frac{v_{max}-v_{min}}{y_{max}-y_{min}} & \frac{v_{max}-v_{min}}{y_{max}-y_{min}}(-y_{min}) + v_{min} \\ 0 & 0 & 1 \end{bmatrix}$$