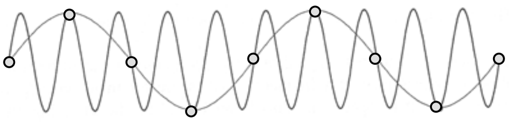
Aliasing, Image Sampling and Reconstruction



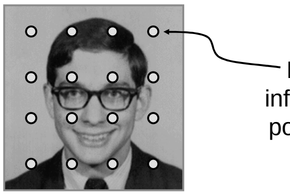
Many of the slides are taken from Thomas Funkhouser course slides and the rest from various sources over the web…

Recall: a pixel is a point…

- It is NOT a box, disc or teeny wee light
- It has no dimension
- It occupies no area
- It can have a coordinate
- *More* than a point, it is a *SAMPLE*

Image Sampling

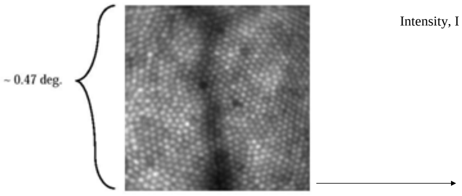
- An image is a 2D rectilinear array of samples
  - Quantization due to limited intensity resolution
  - Sampling due to limited spatial and temporal resolution



Pixels are infinitely small point samples

Imaging devices area sample.

- In video camera the CCD array is an area integral over a pixel.
- The eye: photoreceptors



J. Liang, D. R. Williams and D. Miller, "Supernormal vision and high- resolution retinal imaging through adaptive optics," J. Opt. Soc. Am. A 14, 2884- 2892 (1997)

Sampling and Reconstruction

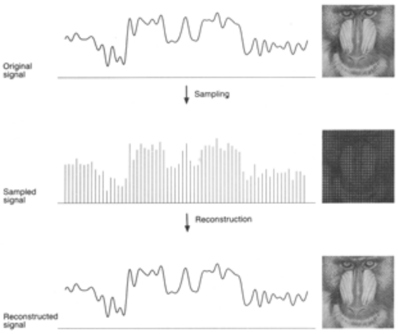
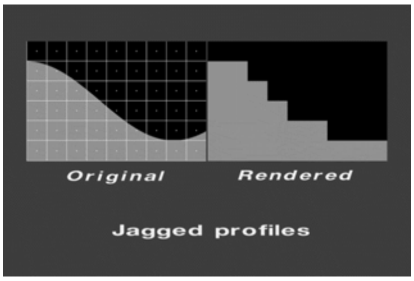


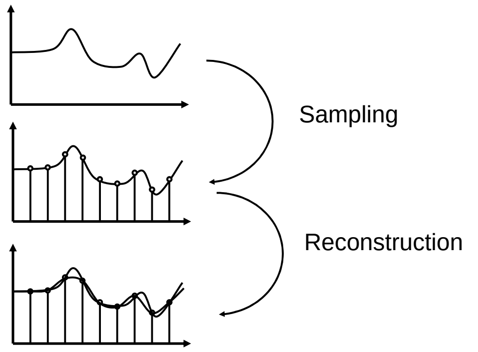
Figure 19.9 FvDFH

Reconstruction artefact



Slide © Rosalee Nerheim-Wolfe

Sampling and Reconstruction



Sampling

Reconstruction

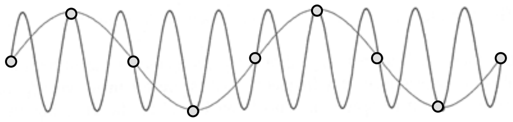
Sources of Error

- Intensity quantization
  - Not enough intensity resolution
- Spatial aliasing
  - Not enough spatial resolution
- Temporal aliasing
  - Not enough temporal resolution

$$E^2 = \sum_{(x,y)} \left(I(x,y) - P(x,y)\right)^2$$

Aliasing (in general)

- In general:
  - Artifacts due to under-sampling or poor reconstruction
- Specifically, in graphics:
  - Spatial aliasing
  - Temporal aliasing



Under-sampling

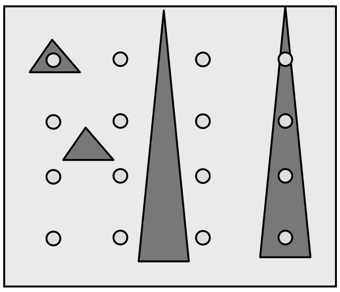
Figure 14.17 FvDFH

Sampling & Aliasing

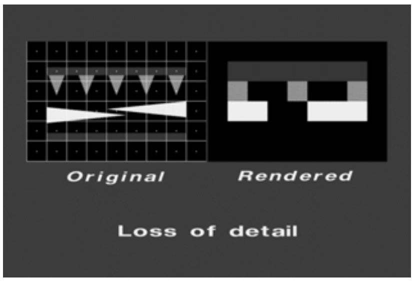
- Real world is continuous
- The computer world is discrete
- Mapping a continuous function to a discrete one is called <u>sampling</u>
- Mapping a continuous variable to a discrete one is called <u>quantizaion</u>
- To represent or render an image using a computer, we must both sample and quantize

Spatial Aliasing

- Artifacts due to limited spatial resolution



Can be a serious problem…



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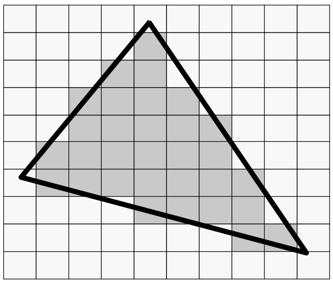
Spatial Aliasing

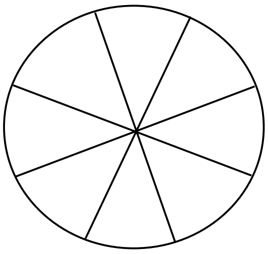
- Artifacts due to limited spatial resolution



“Jaggies”

Temporal Aliasing

- Artifacts due to limited temporal resolution
  - Strobing
  - Flickering



Temporal Aliasing

- Artifacts due to limited temporal resolution
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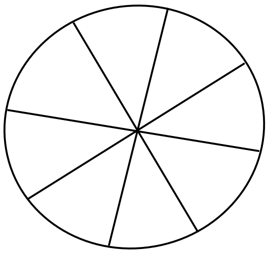
  - Flickering



Temporal Aliasing

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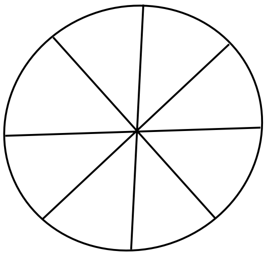

Temporal Aliasing

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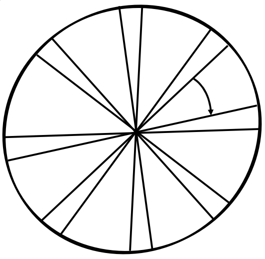
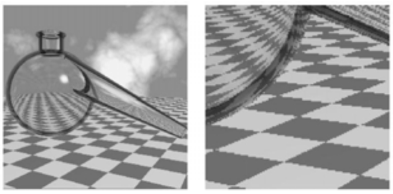


Staircasing or *Jaggies*

The raster *aliasing* effect – removal is called *antialiasing*

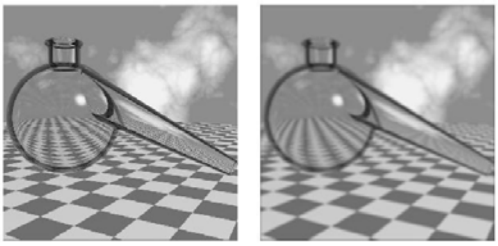


Images by Don Mitchell

Nearest neighbor sampling

Filtered Texture:

Blurring doesn’t work well.



Removed the *jaggies*, but also all the detail ! → Reduction in resolution

Unweighted Area Sampling

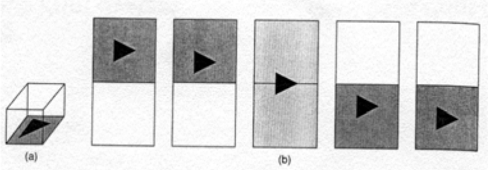


Fig. 14.11 Unweighted area sampling. (a) All points in the pixel are weighted equally. (b) Changes in computed intensities as an object moves between pixels.

Weighted Area Sampling

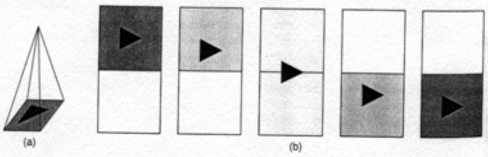


Fig. 14.12 Weighted area sampling. (a) Points in the pixel are weighted differently. (b) Changes in computed intensities as an object moves between pixels.

…with Overlap

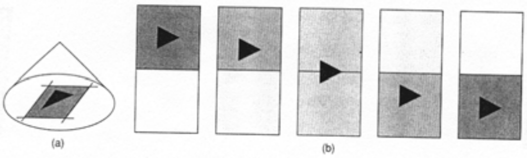


Fig. 14.13 Weighted area sampling with overlap. (a) Typical weighting function. (b) Changes in computed intensities as an object moves between pixels.

Sampling and Reconstruction

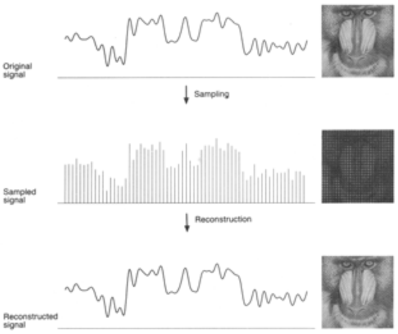


Figure 19.9 FvDFH

Antialiasing

- Sample at higher rate
  - Not always possible
  - Doesn't always solve problem
- Pre-filter to form bandlimited signal
  - Form bandlimited function (low-pass filter)
  - Trades aliasing for blurring

Must consider sampling theory!

How is antialiasing done?

- We need some mathematical tools to
  - analyse the situation.
  - find an optimum solution.
- Tools we will use :
  - Fourier transform.
  - Convolution theory.
  - Sampling theory.

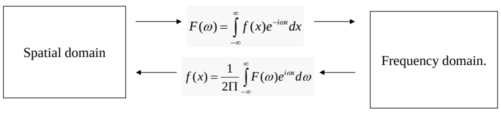
We need to understand the behavior of the signal in frequency domain

Spectral Analysis / Fourier Transforms

- Spectral representation treats the function as a weighted sum of sines and cosines
- Every function has two representations
  - Spatial (time) domain - normal representation
  - Frequency domain - spectral representation
- The *Fourier transform* converts between the spatial and frequency domains.

Spectral Analysis / Fourier Transforms

Note the Euler formula : $e^{it} = \cos t + i \sin t$



Spatial domain

$$F(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$$

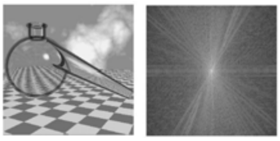
$$f(x) = \frac{1}{2\Pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega x} d\omega$$

Frequency domain.

- The *Fourier transform* converts between the spatial and frequency domain.
- Real and imaginary components.
- Forward and reverse transforms very similar.

Fourier transform conventions.

- We will use the 'optical' convention.

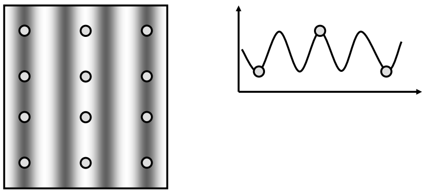


Note : spectral transform has origin in centre, and is symmetrical

Low frequencies in centre,
High frequencies at the edge. Note symmetry.

Sampling Theory

- How many samples are required to represent a given signal without loss of information?
- What signals can be reconstructed without loss for a given sampling rate?



Spectral Analysis

- Spatial domain:
  - Function: f(x)
  - Filtering: convolution
- Frequency domain:
  - Function: F(u)
  - Filtering: multiplication

Any signal can be written as a sum of periodic functions.

1-D Fourier Transform

$$F(u) = \frac{1}{2\pi}\int I(x)\exp(-jux)dx$$

$$I(x) = \frac{1}{2\pi}\int F(u)\exp(jux)du$$

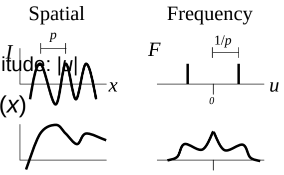
$$j^2 = -1$$

$$\exp(-jux) = \cos ux - j\sin ux$$

$$|a+bj| = \sqrt{a^2+b^2}$$

$$\arg a+bj = \arctan(b/a)$$

- Makes any signal *I*(*x*) out of sine waves
- Converts spatial domain into frequency domain
- Yields spectrum *F*(*u*) of frequencies *u*
  - *u* is actually complex
  - Only worried about amplitude: |*u*|
- DC term: *F*(0) = mean *I*(*x*)
- Symmetry: *F*(-*u*) = *F*(*u*)



Fourier Transform

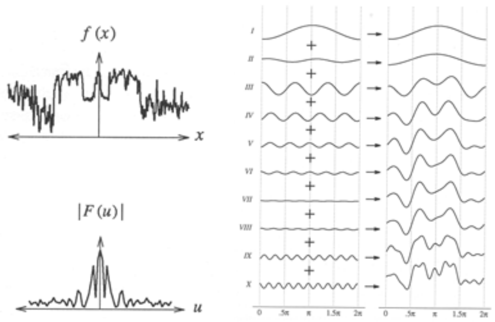


Figure 2.6 Wolberg

Fourier Transform

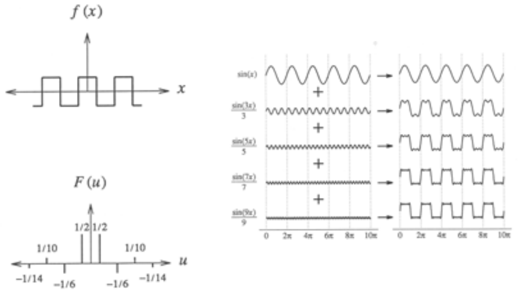


Figure 2.5 Wolberg

Sampling Theorem

- A signal can be reconstructed from its samples, if the original signal has no frequencies above 1/2 the sampling frequency - Shannon
- The minimum sampling rate for bandlimited function is called "Nyquist rate"

A signal is bandlimited if its
highest frequency is bounded.
The frequency is called the bandwidth.

Convolution

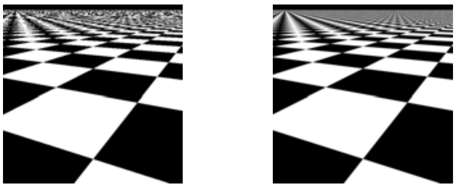
- Convolution of two functions (= filtering):

$$g(x) = f(x) \otimes h(x) = \int_{-\infty}^{\infty} f(\lambda)h(x-\lambda)d\lambda$$

- Convolution theorem
  - Convolution in frequency domain is same as multiplication in spatial domain, and vice-versa

Antialiasing in Image Processing

- General Strategy
  - Pre-filter transformed image via convolution with low-pass filter to form bandlimited signal
- Rationale
  - Prefer blurring over aliasing



Filtering in the frequency domain

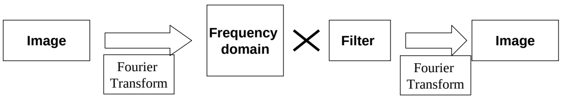
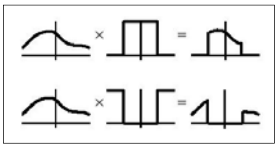


Image → Fourier Transform → **Frequency domain** × **Filter** → Fourier Transform → **Image**

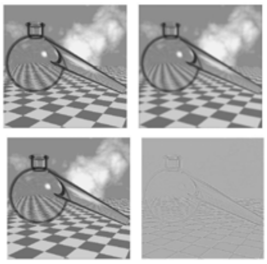


Lowpass filter

Highpass filter

Low and High Pass Filtering.

- Low pass
- High pass



Low-pass Filtering

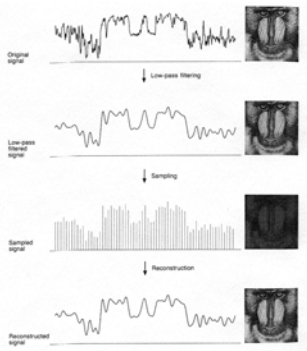


Fig. 14.20 The sampling pipeline with filtering. (Courtesy of George Wolberg, Columbia University.)

Low-pass Filtering

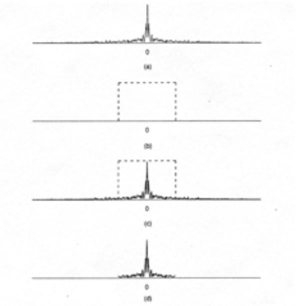


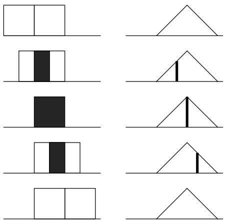
Fig. 14.21 Low-pass filtering in the frequency domain. (a) Original spectrum. (b) Low-pass filter. (c) Spectrum with filter. (d) Filtered spectrum. (Courtesy of George Wolberg, Columbia University.)

Product and Convolution

- Product of two functions is just their product at each point
- Convolution is the sum of products of one function at a point and the other function at all other points
- E.g. Convolution of square wave with square wave yields triangle wave
- Convolution in spatial domain is product in frequency domain, and vice versa

$$(gh)(x) = g(x)h(x)$$

$$(g * h)(x) = \int g(s)h(x-s)ds$$

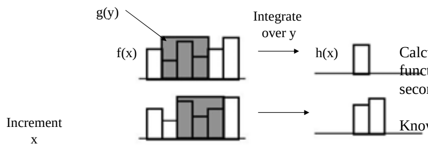


f*g ⇔ FG

fg ⇔ F*G

Filtering in the space domain

- Blurring or averaging pixels together.



Calculate integral of one function, f(x) by a sliding second function g(x-y).

Known as *Convolution.*

$$h(x) = f \otimes g = \int f(x)g(x-y)dy$$

Low-pass Filtering

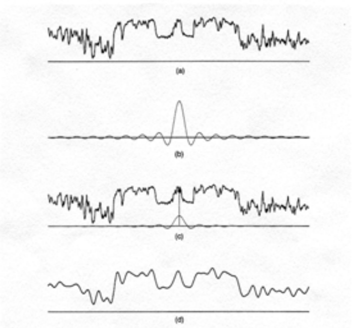
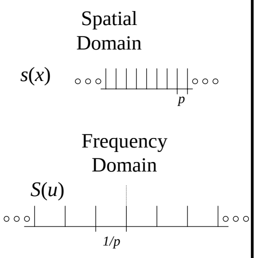


Fig. 14.23 Low-pass filtering in the spatial domain. (a) Original signal. (b) Sinc filter. (c) Signal with filter, with value of filtered signal shown as a black dot (●) at filter's origin. (d) Filtered signal. (Courtesy of George Wolberg, Columbia University.)

Sampling Functions

- Sampling takes measurements of a continuous function at discrete points
- Equivalent to product of continuous function and sampling function
- Uses a sampling function $s(x)$
- Sampling function is a collection of spikes
- Frequency of spikes corresponds to their resolution
- Frequency is inversely proportional to the distance between spikes
- Fourier domain also spikes
- Distance between spikes is the frequency



Sampling , the Comb function

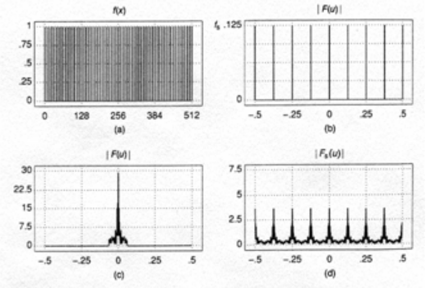
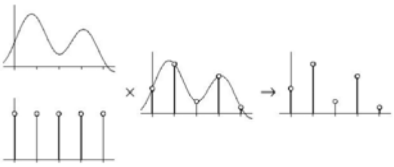


Fig. 14.26 (a) Comb function and (b) its Fourier transform. Convolving the comb's Fourier transform with (c) a signal's Fourier transform in the frequency domain yields (d) the replicated spectrum of the sampled signal. (Courtesy of George Wolberg, Columbia University.)

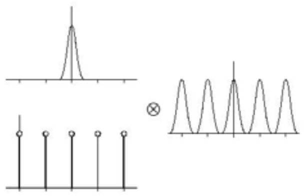
How can we represent sampling ?

Multiplication of the sample with a regular train of delta functions.



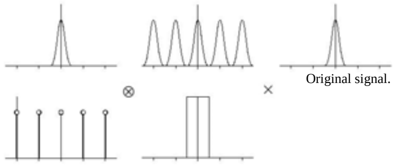
Sampling: Frequency domain.

The Fourier transform of regular comb of delta functions is a comb.
Spacing is inversely proportional



Multiple solutions at regularly increasing values of *f*

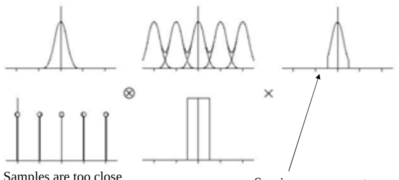
Reconstruction in frequency domain



Original signal.

Bandpass filter due to regular array of pixels.

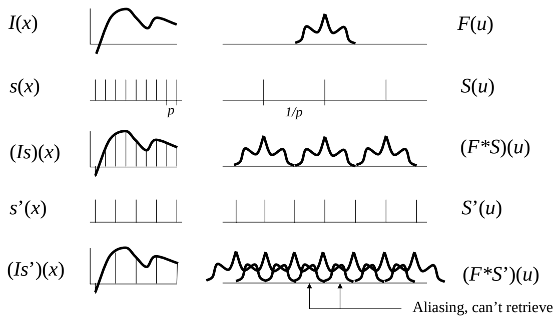
Undersampling leads to aliasing.



Samples are too close together in f.

Spurious components : Cause of aliasing.

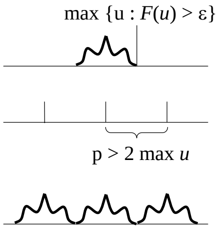
Sampling

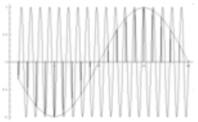


Aliasing, can't retrieve original signal

Shannon's Sampling Theorem

- Sampling frequency needs to be at least twice the highest signal frequency
- Otherwise the first replica interferes with the original spectrum
- Sampling below this *Nyquist limit* leads to aliasing
- Conceptually, need one sample for each peak and another for each valley





The Sampling Theorem.

This result is known as the *Sampling Theorem*
and is due to Claude Shannon who first discovered it in 1949

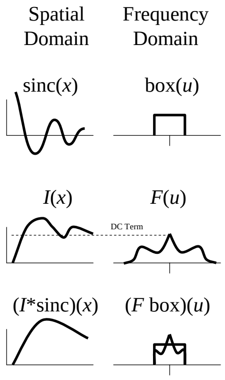
A signal can be reconstructed from its samples without loss of information, if the original signal has no frequencies above 1/2 the sampling frequency

For a given bandlimited function, the rate at which it must be sampled is called the *Nyquist Frequency*

Prefiltering

- Aliases occur at high frequencies
  - Sharp features, edges
  - Fences, stripes, checkerboards
- Prefiltering removes the high frequency components of an image before it is sampled
- Box filter (in frequency domain) is an ideal low pass filter
  - Preserves low frequencies
  - Zeros high frequencies
- Inverse Fourier transform of a box function is a sinc function

  $\mathrm{sinc}(x) = \sin(x)/x$
- Convolution with a sinc function removes high frequencies



Spatial Domain

Frequency Domain

$\mathrm{sinc}(x)$

$\mathrm{box}(u)$

$I(x)$

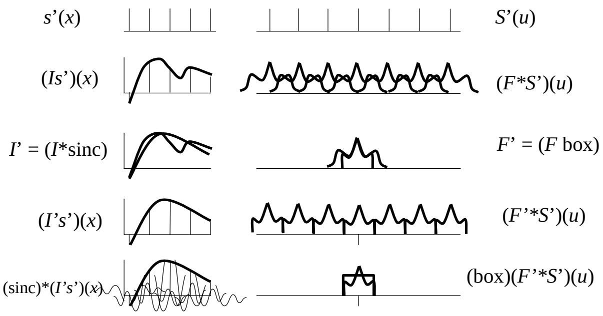
$F(u)$

DC Term

$(I * \mathrm{sinc})(x)$

$(F\ \mathrm{box})(u)$

Prefiltering Can Prevent Aliasing



$s'(x)$

$S'(u)$

$(Is')(x)$

$(F*S')(u)$

$I' = (I*\mathrm{sinc})$

$F' = (F\ \mathrm{box})$

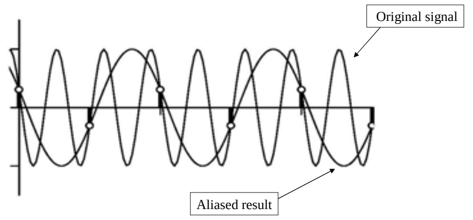
$(I's')(x)$

$(F'*S')(u)$

$(\mathrm{sinc})*(I's')(x)$

$(\mathrm{box})(F'*S')(u)$

Aliasing in the space domain.



Original signal

Aliased result

Summary : Aliasing is the appearance of spurious signals when the frequency of the input signal goes above the Nyquist limit.

Sampling at the Nyquist Frequency

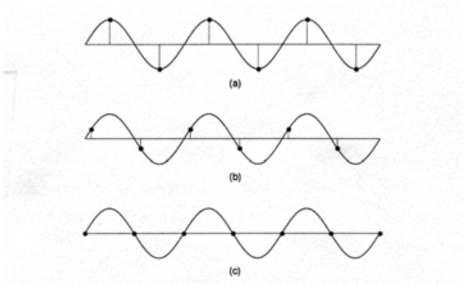


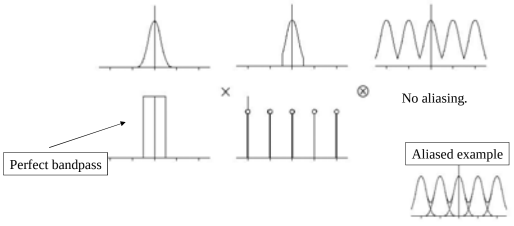
Fig. 14.16 Sampling at the Nyquist frequency (a) at peaks, (b) between peaks, (c) at zero crossings. (Courtesy of George Wolberg, Columbia University.)

Sampling Below the Nyquist Frequency

Fig. 14.17 Sampling below the Nyquist rate. (Courtesy of George Wolberg, Columbia University.)

How do we remove aliasing ?

- Perfect solution - prefilter with perfect bandpass filter.



No aliasing.

Perfect bandpass

Aliased example

How do we remove aliasing ?

- Perfect solution - prefilter with perfect bandpass filter.
  - Difficult/Impossible to do in frequency domain.

- Convolve with sinc function in space domain
  - Optimal filter - better than area sampling.
  - Sinc function is infinite !!
  - Computationally expensive.

How do we remove aliasing ?

- Cheaper solution : take multiple samples for each pixel and average them together → supersampling.
- Can weight them towards the centre → weighted average sampling
- Stochastic sampling

Removing aliasing is called *antialiasing*

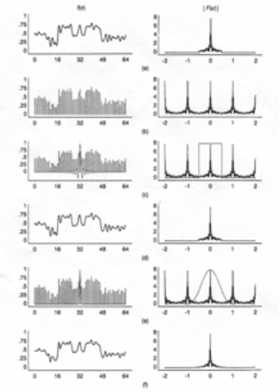


Fig. 14.27 Sampling and reconstruction: Adequate sampling rate. (a) Original signal. (b) Sampled signal. (c) Sampled signal ready to be reconstructed with sinc. (d) Signal reconstructed with sinc. (e) Sampled signal ready to be reconstructed with triangle. (f) Signal reconstructed with triangle. (Courtesy of George Wolberg, Columbia University.)

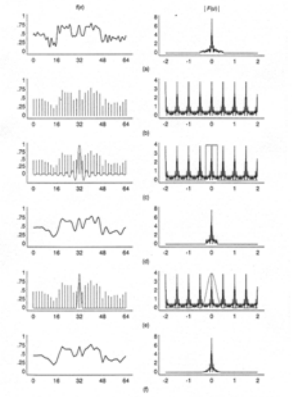


Fig. 14.28 Sampling and reconstruction: Inadequate sampling rate. (a) Original signal. (b) Sampled signal. (c) Sampled signal ready to be reconstructed with sinc. (d) Signal reconstructed with sinc. (e) Sampled signal ready to be reconstructed with triangle. (f) Signal reconstructed with triangle. (Courtesy of George Wolberg, Columbia University.)

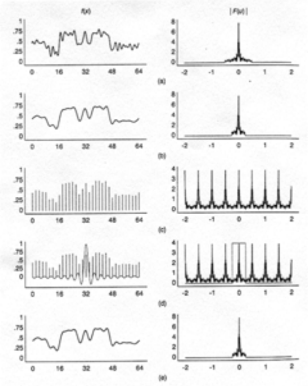


Fig. 14.29 Filtering, sampling, and reconstruction: Sampling rate is adequate after filtering. (a) Original signal. (b) Low-pass filtered signal. (c) Sampled signal. (d) Sampled signal ready to be reconstructed with sinc. (e) Signal reconstructed with sinc. (Courtesy of George Wolberg, Columbia University.)

The 'Sinc' function.

$$\int_{-\infty}^{\infty} square(x) e^{-i\omega x} dx = \int_{-1/2}^{1/2} e^{-i\omega x} dx$$

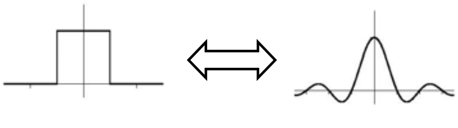
$$= \left. \frac{e^{-i\omega x}}{-i\omega x} \right|_{-\frac{1}{2}}^{\frac{1}{2}}$$

$$= \frac{e^{-\frac{1}{2}i\omega} - e^{\frac{1}{2}i\omega}}{\frac{1}{2}i\omega + \frac{1}{2}i\omega} = \frac{\sin \frac{1}{2}\omega}{\frac{1}{2}\omega}$$

= sinc f

Recall Euler's formula :

$$e^{it} = \cos t + i \sin t$$



The Sinc Filter

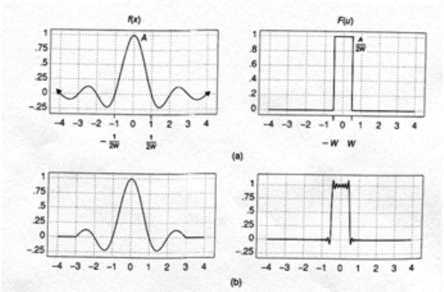


Fig. 14.24 (a) Sinc in spatial domain corresponds to pulse in frequency domain. (b) Truncated sinc in spatial domain corresponds to ringing pulse in frequency domain. (Courtesy of George Wolberg, Columbia University.)

Common Filters

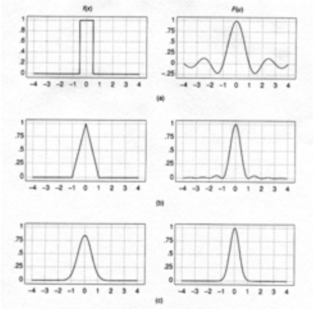


Fig. 14.25 Filters in spatial and frequency domains. (a) Pulse—sinc. (b) Triangle—sinc². (c) Gaussian—Gaussian. (Courtesy of George Wolberg, Columbia University.)

Sample-and-Hold

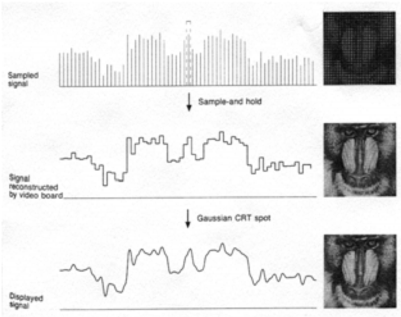
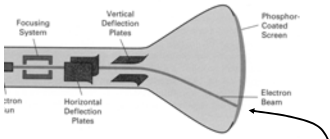


Fig. 14.30 Reconstruction by sample and hold and Gaussian CRT spot. (Courtesy of George Wolberg, Columbia University.)

Image Reconstruction

- Re-create continuous image from samples
  - Example: cathode ray tube



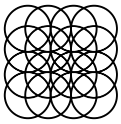


Image is reconstructed
by displaying pixels
with finite area
(Gaussian)

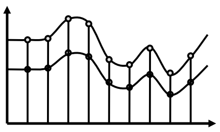
End…

Adjusting Brightness

- Simply scale pixel components
  - Must clamp to range (e.g., 0 to 255)

Original

Brighter



Adjusting Contrast

- Compute mean luminance $\mathcal{L}$ for all pixels
  - luminance = 0.30*r + 0.59*g + 0.11*b
- Scale deviation from $\mathcal{L}$ for each pixel component
  - Must clamp to range (e.g., 0 to 255)

Original

More Contrast

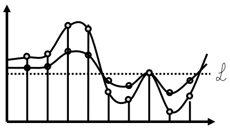
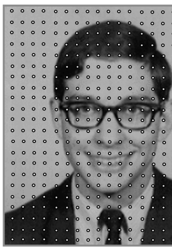


Image Processing

- Consider reducing the image resolution

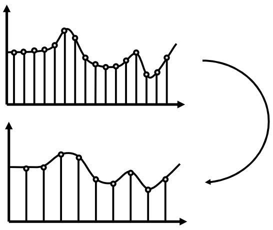


Original image

1/4 resolution

Image Processing

- Image processing is a resampling problem



Resampling

Thou shalt avoid aliasing!

Image Processing

- Quantization
  - Uniform Quantization
  - Random dither
  - Ordered dither
  - Floyd-Steinberg dither
- Pixel operations
  - Add random noise
  - Add luminance
  - Add contrast
  - Add saturation
- Filtering
  - Blur
  - Detect edges
- Warping
  - Scale
  - Rotate
  - Warps
- Combining
  - Morphs
  - Composite

Adjust Blurriness

- Convolve with a filter whose entries sum to one
  - Each pixel becomes a weighted average of its neighbors

Original

Blur

$$\text{Filter} = \begin{bmatrix} 1/16 & 2/16 & 1/16 \\ 2/16 & 4/16 & 2/16 \\ 1/16 & 2/16 & 1/16 \end{bmatrix}$$

Edge Detection

- Convolve with a filter that finds differences between neighbor pixels

Original

Detect edges

$$\text{Filter} = \begin{bmatrix} -1 & -1 & -1 \\ -1 & +8 & -1 \\ -1 & -1 & -1 \end{bmatrix}$$

Image Processing

- Quantization
  - Uniform Quantization
  - Random dither
  - Ordered dither
  - Floyd-Steinberg dither
- Pixel operations
  - Add random noise
  - Add luminance
  - Add contrast
  - Add saturation
- Filtering
  - Blur
  - Detect edges
- Warping
  - Scale
  - Rotate
  - Warps
- Combining
  - Morphs
  - Composite

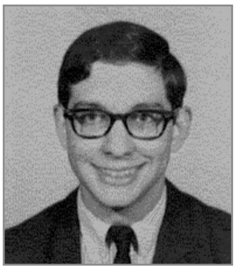
Scaling

- Resample with triangle or Gaussian filter

More on this next lecture!


Original

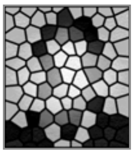

1/4X
resolution


4X
resolution

Summary

- Image processing is a resampling problem
  - Avoid aliasing
  - Use filtering




Triangle Filter

- Convolution with triangle filter

- Convolution with Gaussian filter

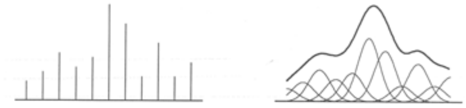


Figure 2.4 Wolberg