EFFICIENTLY DECODING REED MULLER CODES FROM RANDOM ERRORS

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Joint work with

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Amir Shpilka

Tel Aviv University

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This talk is about decoding **Reed-Muller codes** from **random** errors.

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- **Rate:** dimension/block length $= \binom{m}{\leq r} / 2^m$

MORE PROPERTIES OF REED-MULLER CODES

Generator Matrix: evaluation matrix of $\deg \le r$ monomials:

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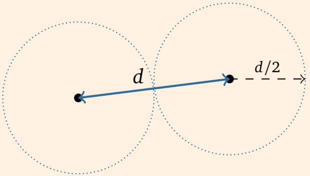
DECODING REED-MULLER CODES

Worst Case Errors: Up to $d/2$ (d is minimal distance).



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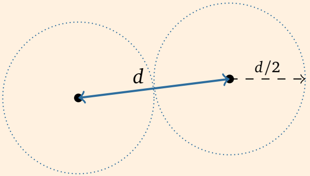

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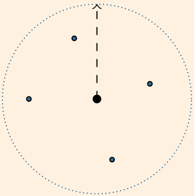

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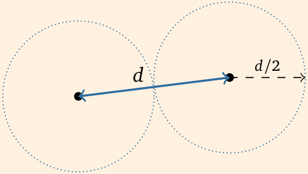

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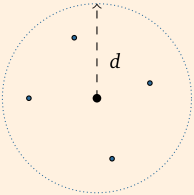

Gopalan-Klivans-Zuckerman08, Bhowmick-Lovett15:

List decoding radius = d.

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- Standard model in coding theory with recent breakthroughs in the last few years (e.g. Arıkan's polar codes)
- An ongoing research endeavor: how do Reed-Muller perform in Shannon's random error model?

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Shannon48: max rate that enables decoding (w.h.p.) is $1 - p$ (for BEC) and $1 - H(p)$ (for BSC). Codes achieving bound called **capacity achieving**.

Category:Capacity-achieving codes

Help

From Wikipedia, the free encyclopedia

Pages in category "Capacity-achieving codes"

This category contains only the following page. This list may not reflect recent changes (learn more).

P

- Polar code (coding theory)

Categories: Error detection and correction

Decoding Erasures

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- **Question:** Can we correct $(1-o(1))\binom{m}{\leq r}$ erasures?
- if yes, $RM(m, m-r-1)$ **achieves capacity** for the BEC

REED-MULLER CODES AND THE BEC

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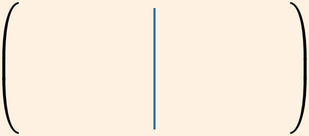

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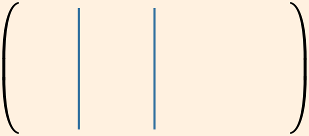

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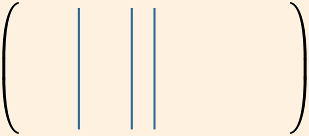

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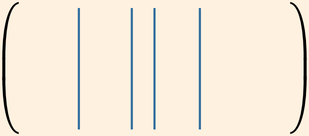

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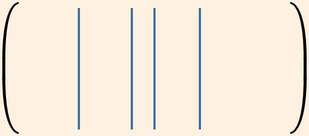

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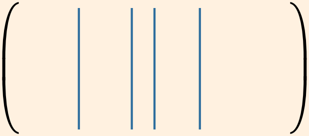

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sanity check: $r = 1$ is good

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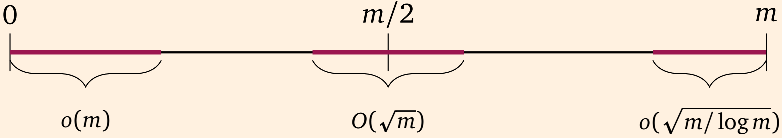

REED-MULLER CODES AND THE BEC

Main Question: Does $RM(m,r)$ achieve capacity for BEC?

Abbe-Shpilka-Wigderson15: YES if $r = o(m)$ or $r = m - o(\sqrt{m/\log m})$.

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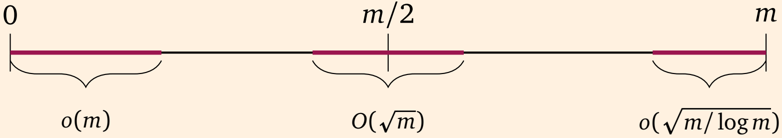

Open Problem: Prove for every degree r.

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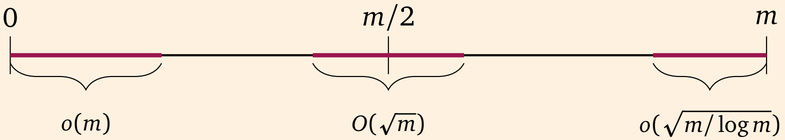

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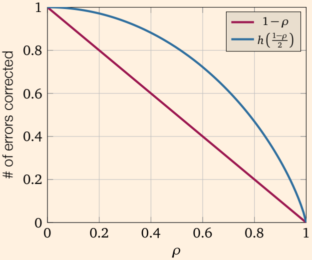

PROOF IDEA

Goal: decode in $RM(m, m-2r-2)$ every pattern which is correctable from erasures in $RM(m, m-r-1)$.

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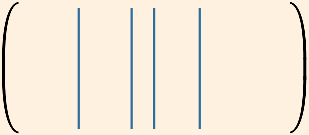
recall: $\{\mathbf{u}_1, \ldots, \mathbf{u}_t\}$ correctable from erasures iff $\{\mathbf{u}_1^r, \ldots, \mathbf{u}_t^r\}$ are linearly independent.



$E(m,r)$

DUAL POLYNOMIALS

Fact: If $\{\mathbf{u}_1^r, \ldots, \mathbf{u}_t^r\}$ lin. indep., $\exists$ polys $\{f_1, \ldots, f_t\}$ of $\deg \leq r$ such that

$$f_i(\mathbf{u}_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j. \end{cases}$$

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Our approach would be to **find** those polynomials.

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If U lin. indep. and $\mathbf{v} = \mathbf{u}_i \in U$, f_i is a solution. Conversely, if solvable and U lin. indep., can show $\mathbf{v} \in U$.

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Corollary: The syndrome of $\mathbf{y}$ is a $\binom{m}{\leq 2r+1}$ long vector α, where $\alpha_M = \sum_{i=1}^{t} M(\mathbf{u}_i)$, for every monom M, $\deg M \leq 2r+1$.

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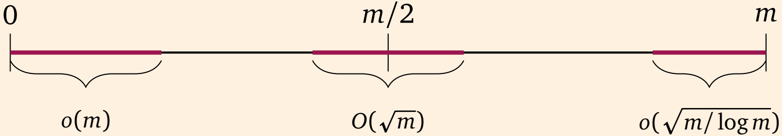

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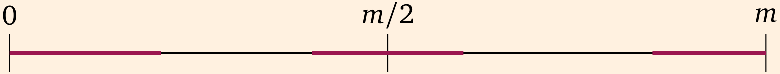

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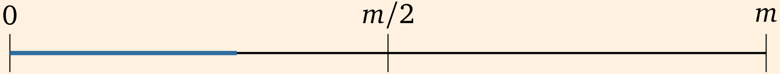

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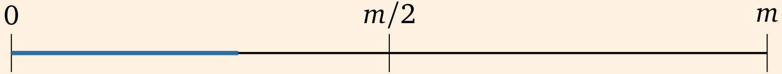

THANK YOU