1. Prove or disprove:
   (a) The class $\mathcal{RE}$ is closed under union and intersection
   (b) The class $\text{co-}\mathcal{RE}$ is closed under union and intersection

2. For the following languages determine whether they belong to $\mathcal{R}$, $\mathcal{RE} \backslash \mathcal{R}$, $\text{co-}\mathcal{RE} \backslash \mathcal{R}$ or none of the above. Prove your claims.
   (a) Input: Turing machine $M$
       Question: is there an $x$ for which $M$ halts?
   (b) Input: Turing machine $M$
       Question: is every even number a sum of two primes (Goldbach’s conjecture, to which the answer is unknown) ?
   (c) Input: Turing machine $M$ and inputs $x$ and $y$
       Question: does $M$ halt on exactly one of the inputs?
   (d) Input: Turing machine $M$
       Question: is $|L(M)| > 3$ ($L(M) = \{\omega | M \text{ accepts } \omega \}$) ?
   (e) Input: Turing machine $M$
       Question: is $|L(M)| \leq 3$?
   (f) Input: Turing machine $M$
       Question: is $L(M) \in \mathcal{R}$?
   (g) Input: Turing machine $M$
       Question: is $L(M) \in \mathcal{RE}$?
   (h) Input: Turing machine $M$ such that $|\langle M \rangle| < 10^{100}$
       Question: does $M$ halt on all inputs?

3. (a) Let $A$ be a decidable language. Show a mapping reduction: $A \leq_m H_{TM}$
       (The halting problem).
(b) Let $\mathcal{M}$ be the set of Turing machines that always halt. Let $L = \{M \in \mathcal{M}|M$ accepts the empty string$\}$ (that is, the input is always a Turing machine that halts). Let $EVEN$ be the languages of even numbers over the binary alphabet. Show a mapping reduction $L \leq_m EVEN$. Why would this not work for Turing machines that don’t necessarily halt?

4. Prove or disprove: if the language $L$ is not decidable and $L \leq_m \overline{L}$, then $L \notin \mathcal{RE}, L \notin \text{co-RE}$.

5. Let $L_1, L_2 \in \mathcal{RE}\backslash\mathcal{R}$. Prove whether the following is possible:

(a) $L_1 \cup L_2 \in \mathcal{R}$

(b) $L_1 \cap L_2 \in \mathcal{R}$

(c) $L_1 \cup L_2 \in \mathcal{R}$ and $L_1 \cap L_2 \in \mathcal{R}$