Computational Models
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Assignment 5 (and last)

Posted: 6/1/05. Due: 20/1/05, 16:00.

1. We say that a Turing machine with encoding $\langle M \rangle$ is *minimal* if no equivalent TM has a strictly shorter encoding. Define

$$MIN = \{\langle M \rangle \mid M \text{ is a minimal Turing machine}\} .$$

   (a) Prove that $MIN \notin RE$.

   (b) (optional) Suppose $L \subseteq MIN$, and L is infinite. Prove that $L \notin RE$.

2. An oracle TM M has a working tape and a query tape, and can ask an oracle O whether the data on the query tape belongs to O or not. For concreteness, let us focus on P^{NP}, the class of all languages accepted by an oracle TM, M, having access to a SAT oracle. M is a deterministic polynomial time machine, and the queries it makes are whether the strings q it writes down on the query tape are satisfiable formulae (i.e. $q \in SAT$) or not (i.e. $q \notin SAT$). One way to think about it is that during the computation on input w, the machine M can ask a clairvoyant, O, whether a formula it wrote down is satisfiable or not. The clairvoyant always answers correctly, and charges just one step for her services (M is also charged for writing down the formula the number of steps the writing took.)

   Prove the following:

   (a) $NP \subseteq P^{NP}$.

   (b) $coNP \subseteq P^{NP}$.

   (c) P^{NP} is closed under complement.

   (d) The $MaxClick$ problem consists of pairs (G, k) where $G = (V, E)$ is an undirected graph, and k is the size of the largest click(s) in G.
   Prove that $MaxClick \in P^{NP}$.

3. Prove that the halting problem ($HALT$) is NP-Hard.

4. We say that a language L is in $AvTime(T(n))$ if there is a deterministic Turing machine solving L, whose average running time over all inputs of size n is at most $T(n)$. We denote $AvP = \bigcup_{c>0} AvTime(n^c)$. We also denote $E = \bigcup_{c>0} Time(2^{cn})$.

   Prove that $P \subseteq AvP \subseteq E$.

5. We say that a non-deterministic Turing machine is *nice* if for every input x the following holds:

   - Every computation path returns either 'accept', 'reject' or 'quit'.
   - There is at least one non-quit path.
   - All non-quit paths have the same value.

Let $NICE$ be the class of all languages L that are accepted by some non-deterministic, polynomial time, nice machine.

Prove that $NICE = NP \cap coNP$.

6. Given an (un)weighted connected graph $G = (V, E)$, a *spanning tree* of G is a set of vertices $T \subseteq E$, such that T connects all the vertices of G, and there are no cycles in T (namely T is a tree). The $k - SpanTree$ problem consists of all pairs (G, k) where $G = (V, E)$ is an undirected graph that has a spanning tree with at most k leaves.

   Prove that the $k - SpanTree$ problem is NP-Complete.

7. (from last year's exam) Prove that the following problem L is NP-complete:

   Input: An undirected graph $G = (V, E)$, and a positive integer k. The pair (G, k) is in L if L has two *disjoint* sets of vertices, each of size k, such that both sets are independent.

8. (from last year's exam) A *quadratic equation* has 0 on its right hand side, and its left hand side is a sum of terms. Each term is either an integer, an integer times a variable, or an integer times two variables. For example,

$$x_1x_2 + 4x_2x_7 + x_3 + 5x_4^2 + 3x_8 + 6x_2 - 3x_4^2 - 1 = 0 \ .$$

   Prove that the following language is NP-complete:

   $QE = \{$ A system of m quadratic equations over n variables $x_1, ..., x_n \mid$ there is an assignment $x_1, ..., x_n \in \{0, 1\}$ that satisfies all m equations.$\}$

   Hint: Reduction from 3-colorability.