Assignment 5 (and last)
Posted: 6/1/05. Due: 20/1/05, 16:00.

1. We say that a Turing machine with encoding $\langle M \rangle$ is *minimal* if no equivalent TM has a strictly shorter encoding. Define

$$MIN = \{ \langle M \rangle \mid M \text{ is a minimal Turing machine} \} .$$

(a) Prove that $MIN \notin RE$.
(b) (optional) Suppose $L \subseteq MIN$, and $L$ is infinite. Prove that $L \notin RE$.

2. An oracle TM $M$ has a working tape and a query tape, and can ask an oracle $O$ whether the data on the query tape belongs to $O$ or not. For concreteness, let us focus on $P^{NP}$, the class of all languages accepted by an oracle TM, $M$, having access to a SAT oracle. $M$ is a deterministic polynomial time machine, and the queries it makes are whether the strings $q$ it writes down on the query tape are satisfiable formulae (i.e. $q \in SAT$) or not (i.e. $q \notin SAT$). One way to think about it is that during the computation on input $w$, the machine $M$ can ask a clairvoyant, $O$, whether a formula it wrote down is satisfiable or not. The clairvoyant always answers correctly, and charges just one step for her services ($M$ is also charged for writing down the formula the number of steps the writing took.) Prove the following:

(a) $NP \subseteq P^{NP}$.
(b) $coNP \subseteq P^{NP}$.
(c) $P^{NP}$ is closed under complement.
(d) The MaxClick problem consists of pairs $(G, k)$ where $G = (V, E)$ is an undirected graph, and $k$ is the size of the largest click(s) in $G$. Prove that $MaxClick \in P^{NP}$.

3. Prove that the halting problem (HALT) is NP-Hard.

4. We say that a language $L$ is in $AvTime(T(n))$ if there is a deterministic Turing machine solving $L$, whose average running time over all inputs of size $n$ is at most $T(n)$. We denote $AvP = \bigcup_{c>0} AvTime(2^{cn})$. We also denote $E = \bigcup_{c>0} Time(2^{cn})$.

Prove that $P \subseteq AvP \subseteq E$.

5. We say that a non-deterministic Turing machine is *nice* if for every input $x$ the following holds:

- Every computation path returns either ‘accept’, ‘reject’ or ‘quit’.
- There is at least one non-quit path.
- All non-quit paths have the same value.
Let $NICE$ be the class of all languages $L$ that are accepted by some non-deterministic, polynomial time, nice machine.
Prove that $NICE = NP \cap coNP$.

6. Given an (un)weighted connected graph $G = (V, E)$, a spanning tree of $G$ is a set of vertices $T \subseteq E$, such that $T$ connects all the vertices of $G$, and there are no cycles in $T$ (namely $T$ is a tree). The $k - SpanTree$ problem consists of all pairs $(G, k)$ where $G = (V, E)$ is an undirected graph that has a spanning tree with at most $k$ leaves.
Prove that the $k - SpanTree$ problem is NP-Complete.

7. (from last year's exam) Prove that the following problem $L$ is NP-complete:
   **Input:** An undirected graph $G = (V, E)$, and a positive integer $k$. The pair $(G, k)$ is in $L$ if $L$ has two disjoint sets of vertices, each of size $k$, such that both sets are independent.

8. (from last year’s exam) A quadratic equation has 0 on its right hand side, and its left hand side is a sum of terms. Each term is either an integer, an integer times a variable, or an integer times two variables. For example,
   $$x_1 x_2 + 4x_2 x_7 + x_3 + 5x_4^2 + 3x_8 + 6x_2 - 3x_4^2 - 1 = 0 .$$
Prove that the following language is NP-complete:

   $QE = \{ \text{A system of } m \text{ quadratic equations over } n \text{ variables } x_1, ..., x_n \mid \text{there is an assignment } x_1, ..., x_n \in \{0, 1\} \text{ that satisfies all } m \text{ equations.} \}$
   **Hint:** Reduction from 3-colorability.