Assignment 3
Handed: 1/12/04, Due: 14/12/04, no later than 16:00

1. Write down a description of a TM (with all details, including the transition function $\delta$) that decides the following languages:
   (a) $L_1 = \{w \in \{0, 1\}^* | w$ does not include two consecutive zeros$\}$
   (b) $L_2 = \{a^n b^n c^n | n \in \mathbb{N}\}$

2. Describe in details but in words (no need to explicitly write down the transition function $\delta$) a deterministic TM, that computes the product of two binary numbers. You can use any (finite) number of tapes to make “programming” the TM easier.

3. Prove each of the following statements, or disprove it by giving a counter example.
   (a) $\mathcal{R}$ is closed under infinite union, that is, if $L_1, L_2, \ldots \in \mathcal{R}$ then $\bigcup_{i=1}^{\infty} L_i \in \mathcal{R}$.
   (b) $\mathcal{R}$ is closed under infinite intersection.
   (c) $\mathcal{RE}$ is closed under infinite union.
   (d) $\mathcal{RE}$ is closed under infinite intersection.

4. Prove or disprove the existence of two languages $L_1, L_2 \subseteq \{0, 1\}^*$ such that
   (a) $L_1 \cap L_2 \in \mathcal{R}$.
   (b) $L_1 \cup L_2 \in \mathcal{R}$.
   (c) $L_1 \cap L_2 \in \mathcal{R}$ and $L_1 \cup L_2 \in \mathcal{R}$.

5. The accepting language $A_{TM} = \{< M, w > | M$ is a TM that accepts $w\}$ was defined in class and shown to be in $\mathcal{RE} \setminus \mathcal{R}$. Prove that $L \in \mathcal{RE}$ if and only if $L \leq_m A_{TM}$.

6. Let $L_1, L_2 \subseteq \{0, 1\}^*$ be $\mathcal{RE}$ languages such that $L_1 \cup L_2 = \{0, 1\}^*$ and $L_1 \cap L_2 \neq \emptyset$. Show that $L_1 \leq_m (L_1 \cap L_2)$.
7. Without using Rice’s Theorem, determine if the following languages are decidable:

(a) Given an encoding, $< M >$, of a Turing machine, it is in the language iff the Goldbach conjecture is true. (Goldbach’s conjecture is that all positive even integers, $n$, can be expressed as the sum of two primes $p + q$, where 1 is also considered a prime here. It is known to hold for all even $n < 10^{16}$. Incidentally, it was proved that every such $n$ can be written as the sum of at most $300,000$ primes, $p_1 + \ldots + p_{300,000}$, an interesting result that unfortunately does not bring us any closer to the conjecture.)

(b) Given an encoding of a Turing machine, $< M >$, it is in the language iff $M$ on the empty input outputs 17.

(c) Given the encoding of two Turing machines, $< M_1, M_2 >$, it is in the language iff $\forall x \in \Sigma^*, M_1(x) = M_2(x)$. (That is, for all inputs, either they both do not halt, or they both halt and return the same output.)

(d) Given an encoding of a Turing machine, $< M >$, and a natural number $n$, the pair is in the language iff $M$ halts on all inputs, and the length of the encoding, $< M >$, is at most $100 + n$.

(e) Given an encoding of a Turing machine, $< M >$, and a natural number $n$, the pair is in the language iff $M$ halts on all inputs, and the length of the encoding, $< M >$, is at most $100 - n$.

(f) Given an encoding of a Turing machine, $< M >$, it is in the language iff either $M$ halts on the empty input, or the length of the encoding, $< M >$, is larger than 100.