

Assignment 5

Handed: 20/1/04, Due: 3/2/04

1. Recall that an oracle TM, M^L , is a TM with a special *query tape*. It can write down any string s on the query tape, and in one additional step get a reply whether $s \in L$ or $s \notin L$. We define the class P^{SAT} as the collection of languages decided by a polynomial time deterministic TM, with access to an oracle for SAT.

Prove the following:

- (a) $NP \subseteq P^{SAT}$.
 - (b) $coNP \subseteq P^{SAT}$.
 - (c) P^{SAT} is closed under complement.
 - (d) Show that if $NP = P^{SAT}$ then $NP = coNP$.
 - (e) Define $MaxClique := \{(G, k) \mid \text{the size of the largest clique in } G \text{ equals } k\}$.
Prove that $MaxClique \in P^{SAT}$.
2. Prove that the language $NA_{TM} = \{(\langle M \rangle, w) \mid M \text{ is a non-deterministic TM that accepts } w\}$ is NP-Hard.
 3. We say that a language L is in $AvTime(T(n))$ if there is a deterministic Turing machine solving L , whose average running time over all inputs of size n , is at most $T(n)$. When defining the average, assume $\Sigma = \{0, 1\}$, and each string of length n is weighted $\frac{1}{2^n}$. We denote $AvP = \bigcup_{c>0} AvTime(n^c)$. We also denote $E = \bigcup_{c>0} Time(2^{cn})$.
Prove that $P \subseteq AvP \subseteq E$.
 4. We say that a non-deterministic Turing machine is *nice* if for every input x the following holds:
 - Every computation path returns either 'accept', 'reject' or 'quit'.
 - There is at least one non-quit path.
 - All non-quit paths have the same value.

Let $NICE$ be the class of all languages L that are accepted by some nice non-deterministic, polynomial time, Turing machine.

Prove that $NICE = NP \cap coNP$.

5. Given an undirected connected graph $G = (V, E)$, we define a *spanning tree* of G to be a subset $T \subseteq E$, such that T connects all the vertices of G , and there are no cycles in T (T must be a tree). Define $k - SpanTree$ problem to be:
Input: An un-weighted connected graph $G = (V, E)$, and a natural number k .
Question: Is there a spanning tree of G , with at most k leaves?
Prove that $k - SpanTree$ problem is NP-Complete.

6. The problem *MaxEx3SAT* problem is the following optimization problem: The input is a CNF formula ϕ with *exactly* three literals per clause (you can assume that no clause contains both x and $\neg x$). Goal: Find an assignment that satisfies the *maximum* number of clauses.

Suppose ϕ has m clauses. Finding an assignment satisfying all m clauses will solve 3SAT, so we do not really expect to do this in polynomial time. But we can try an *approximation algorithm*.

- (a) Find a polynomial time algorithm that satisfies at least $\frac{m}{2}$ of the clauses (hint: you may use randomization).
- (b) Argue that (a) is an algorithm with approximation ratio $\frac{1}{2}$.
- (c) Improve the approximation ratio to $\frac{7}{8}$.
- (d) To complete the picture, it has been shown (a couple of years ago) that an efficient algorithm with approximation ratio $\frac{7}{8} + \epsilon$ (where $\epsilon > 0$ is a constant) implies P=NP. (No action in this item, unless you want to prove P=NP...)