1. For each of the following languages, determine whether they belong to $R$, and whether they belong to $RE$.

   (a) $L_1 = \{ B \mid B \text{ is a Turing machine and } \exists x. B(x) \neq \bot \}$

   (b) $L_2 = \{ x \mid \text{There is a Turing machine } B, \text{ s.t. } B(x) \neq \bot \}$

   (c) $L_3 = \{ B_1, B_2 \mid B_1, B_2 \text{ are Turing machine and } \exists x. B_1(x) \neq \bot \land B_2(x) \neq \bot \}$

   (d) $L_4 = \{ B_1, B_2 \mid B_1, B_2 \text{ are Turing machine and } \forall x. B_1(x) = B_2(x) \}$

   (e) $L_5 = \{ B \mid B \text{ is a Turing machine and } B \text{ accepts more than one input} \}$

   (f) $L_6 = \{ B \mid B \text{ is a Turing machine and } B \text{ accepts at most one input} \}$

   (g) $L_7 = \{ B \mid B \text{ is a Turing machine and } B(n) = T \iff n \text{ is even} \}$

2. Prove that every infinite and recursively-enumerable language $L$, contains an infinite and recursive language $L' \subseteq L$.

3. A state $q$ of a TM $M$ is called important, if there is at least one input $x$, such that $M$ is in state $q$ at least twice while running on $x$. Otherwise (that is for every $x$, $M$ is in $q$ at most once while running on $x$) $q$ is called not important. Prove or disprove: There is a constant $C$ such that every TM, $M$, is equivalent to a TM, $M'$, with at most $C$ important states. Note that $C$ should be fixed, that is, independent of the TM $M$.

4. For each of the following problems, argue if is it decidable or not.

   (a) Input: TM $M$, and input $x$ for $M$.
       Output: When $M$ runs on $x$, is it in the same state two consecutive steps?

   (b) Input: TM $M$.
       Output: When $M$ runs on the empty string, does its head ever move to the left?

5. Given a TM, $M$, define $F_M$ to be the number of steps that $M$ performs when running on the empty string. If $M$ does not halt on the empty input, define $F_M$ to be 0.

Let $S : \mathbb{N} \to \mathbb{N}$ be a function such that: $S(0) = 0$, and for every $k \geq 1$, $S(k)$ is the maximum of $F_M$ over all $k$ states, one tape deterministic TMs, whose tape alphabet is $\{0, 1\}$ plus the blank symbol.

   (a) Prove that $S(k)$ is monotonically increasing (that is, for all $k \geq 0$, we have $S(k+1) > S(k)$).

   (b) Prove that $S(k)$ is not computable (that is, there does not exist a TM, $M$ such that for all $k \geq 0$, $f_M(k) = S(k)$).
6. (a) Show the existence of a TM, $M_0$, for which the function $K_{M_0}(x)$ is computable. Recall that

$$K_{M_0} = \min_k \{ k : \exists y \ | y | = k \land f_{M_0}(y) = x \}$$

(the Kolmogorov complexity of the string $x$ with respect to $M_0$).

(b) Prove or disprove: the two argument function $f(<M>, x) = K_M(x)$ is computable.

7. A language $L_0$ is called RE-complete if $L_0 \in RE$, and for each $L \in RE$, $L \leq_m L_0$. The notion of R-completeness is defined in similar manner.

(a) Is there a RE-complete language, which belongs to R?

(b) Is there a language $L_0$, which is not in RE, such that for every $L \in RE$, $L \leq_m L_0$?

(c) Prove that if $L_0 \in R$, $L_0 \neq \Sigma^*$, and $L_0 \neq \emptyset$, then $L_0$ is R-complete.

(d) Give an example for a language $L_0$ which belongs to R, and is not R-complete. Prove your claim.