Remarkable Fact (that we want to prove)

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$\Rightarrow$ construct an NFA accepting $R$.

$\Leftarrow$ Given a regular language, $L$, construct an equivalent regular expression.
(⇒⇒) NFA Accepting Reg Expression, $R$

1. $R = a$, for some $a \in \Sigma$
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(⇒⇒) NFA Accepting Reg Expression, \( R \)

\[
R = (R_1 \cup R_2)
\]

\[
R = (R_1 \circ R_2)
\]

\[
R = (R_1)^* 
\]
Example

\[
\begin{align*}
\text{a} & \quad \xrightarrow{a} \quad \circ \quad \circ \\
\text{b} & \quad \xrightarrow{b} \quad \circ \quad \circ \\
\text{ab} & \quad \xrightarrow{b} \quad \circ \quad \xrightarrow{\varepsilon} \quad \circ \quad \xrightarrow{a} \quad \circ \\
\text{ab} \cup a & \quad \xrightarrow{\varepsilon} \quad \circ \quad \xrightarrow{\varepsilon} \quad \circ \quad \xrightarrow{a} \quad \circ
\end{align*}
\]
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- Each transition labeled with a symbol or ε,
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GNFAs are natural generalization of NFAs.
GNFA Special Form

Start state has outgoing arrows to every other state, but no incoming arrows.
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Easy to transform any GNFA into special form.

Really? How? …
Converting DFA to Regular Expression (⇐⇒)

Strategy – sequence of \textit{equivalent} transformations

\begin{itemize}
  \item given a $k$-state DFA
\end{itemize}
Converting DFA to Regular Expression ($\iff$)

Strategy – sequence of equivalent transformations
- given a $k$-state DFA
- transform into $(k + 2)$-state GNFA

Label on single transition is the desired regular expression.
Converting DFA to Regular Expression ($\iff$)

Strategy – sequence of equivalent transformations

- given a $k$-state DFA
- transform into $(k + 2)$-state GNFA
- while GNFA has more than 2 states, transform it into equivalent GNFA with one fewer state
Converting DFA to Regular Expression ($\iff$)

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- given a $k$-state DFA
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Converting Strategy (↔)

3-state DFA

5-state GNFA

4-state GNFA

3-state GNFA

2-state GNFA

regular expression

Slides modified by Benny Chor, based on original slides by Maurice Herlihy, Brown University.
Removing One State

We remove one state $q_r$, and then repair the machine by altering regular expression of other transitions.
Formal Treatment – GNDA Definition

- $q_s$ is start state.
- $q_a$ is accept state.
- $\mathcal{R}$ is collection of regular expressions over $\Sigma$. 
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If $\delta(q_i, q_j) = R$, then arrow from $q_i$ to $q_j$ has label $R$. 
A generalized deterministic finite automaton (GDFA) is $(Q, \Sigma, \delta, q_s, q_a)$, where

- $Q$ is a finite set of states,
- $\Sigma$ is the alphabet,
- $\delta : (Q - \{q_a\}) \times (Q - \{q_s\}) \to \mathcal{R}$ is the transition function.
- $q_s \in Q$ is the start state, and
- $q_a \in Q$ is the unique accept state.
A GNFA accepts a string $w \in \Sigma^*$ if there exists a parsing of $w$, $w = w_1w_2 \cdots w_k$, where each $w_i \in \Sigma^*$, and there exists a sequence of states $q_0, \ldots, q_k$ such that

- $q_0 = q_s$, the start state,
- $q_k = q_a$, the accept state, and
- for each $i$, $w_i \in L(R_i)$, where $R_i = \delta(q_{i-1}, q_i)$.

(namely $w_i$ is an element of the language described by the regular expression $R_i$.)
The CONVERT Algorithm

Given GDFA $G$, convert it to equivalent GNFA $G'$.

1. let $k$ be the number of states of $G$. 
The CONVERT Algorithm

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- Let $Q' = Q - \{q_r\}$.
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  - $R_3 = \delta(q_r, q_j)$, and $R_4 = \delta(q_i, q_j)$. 

Slides modified by Benny Chor, based on original slides by Maurice Herlihy, Brown University.
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- Define $\delta'(q_i, q_j) = (R_1)(R_2)^*(R_3) \cup (R_4)$. 
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- Denote the resulting $k - 1$ states GNFA by $G'$. 
The CONVERT Procedure

We define the recursive procedure CONVERT(·):

Given GDFA $G$.

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- Return \textbf{CONVERT}(G').
Correctness Proof of Construction

Theorem: $G$ and $\text{CONVERT}(G)$ accept the same language.

Proof: By induction on number of states of $G$. 
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Let $G'$ be the $k - 1$ states GNFA produced from $G$ by the algorithm.
$G$ and $G'$ accept the same language

By the induction hypothesis, $G'$ and $\text{CONVERT}(G')$ accept the same language.

Three steps:
1. If $G$ accepts $w$, then so does $G'$.
2. If $G'$ accepts $w$, then so does $G$.
3. Therefore $G$ and $G'$ are equivalent.
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Step One

Claim: If $G$ accepts $w$, then so does $G'$:

- If $G$ accepts $w$, then there exists a “path of states” $q_s, q_1, q_2, \ldots, q_a$ traversed by $G$ on $w$, leading to the accept state $q_a$. 
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- If $q_r$ does not appear on path, then $G'$ accepts $w$ because the new regular expression on each edge of $G'$ contains the old regular expression in the “union part”.

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- If $q_r$ does appear, consider the regular expression corresponding to $\ldots q_i, q_r, \ldots, q_r, q_j \ldots$.

The new regular expression $(R_{i,r})(R_{r,r})^*(R_{r,j})$ linking $q_i$ and $q_j$ encompasses any such string.
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- Either way, the claim holds.
Claim: If $G'$ accepts $w$, then so does $G$.

Proof: Each transition from $q_i$ to $q_j$ in $G'$ corresponds to a transition in $G$, either directly or through $q_r$. Thus if $G'$ accepts $w$, then so does $G$.

This completes the proof of the claim that $L(G) = L(G')$. 
Steps Two and Three

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This completes the proof of the claim that $L(G) = L(G')$.

Combined with the induction hypothesis, this shows that $G$ and the regular expression $\text{CONVERT}(G)$ accept the same language.

This, in turn, proves our remarkable claim: A language, $L$, is described by a regular expression, $R$, if and only if $L$ is regular.
Negative Results

We have made a lot of progress understanding what finite automata can do. But what can’t they do?
We have made a lot of progress understanding what finite automata can do. But what can’t they do? Is there a DFA that accepts

- \( B = \{0^n1^n | n \geq 0\} \)
- \( C = \{w | w \text{ has an equal number of 0’s and 1’s}\} \)
- \( D = \{w | w \text{ has an equal number of occurrences of 01 and 10 substrings}\} \)

Consider \( B \):

- DFA must “remember” how many 0’s it has seen
- impossible with finite state.

The others are exactly the same.
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Question: Is this a proof?

Answer: No, \( D \) is regular!???(see problem set 1)
Pumping Lemma

We will show that all regular languages have a special property.

Suppose $L$ is regular.

If a string in $L$ is longer than a certain critical length $\ell$ (the pumping length),

then it can be “pumped” to a longer string by repeating an internal substring any number of times.

The longer string must be in $L$ too.

This is a powerful technique for showing that a language is not regular.
Pumping Lemma

**Theorem:** If $L$ is a regular language, then there is an $\ell > 0$ (the **pumping length**), where if $s$ is any string in $L$ of length $|s| > \ell$, then $s$ may be divided into three pieces $s = xyz$ such that

1. $|y| > 0$,
2. $|xy| \leq \ell$,
3. for every $i > 0$, $xy^iz \in L$.

**Remarks:** Without the second condition, the theorem would be trivial. The third condition is technical and useful occasionally.
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Pumping Lemma – Proof

Let \( M = (Q, \Sigma, \delta, q_1, F) \) be a DFA that accepts \( L \).

Let \( \ell \) be \( |Q| \), the number of states of \( M \).
Pumping Lemma – Proof

Let $M = (Q, \Sigma, \delta, q_1, F)$ be a DFA that accepts $L$.

Let $\ell$ be $|Q|$, the number of states of $M$.

If $s \in L$ has length at least $\ell$, consider the sequence of states $M$ goes through as it reads $s$: 

\[ s_1 s_2 s_3 s_4 s_5 \ldots s_n \]

Since the sequence of states is of length $|s| + 1 > \ell$, and there are only $\ell$ different states in $Q$, at least one state is repeated (by the pigeonhole principle).
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\[
\begin{array}{cccccccc}
  s_1 & s_2 & s_3 & s_4 & s_5 & s_6 & \ldots & s_n \\
  \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
  q_1 & q_20 & q_9 & q_{17} & q_{12} & q_{13} & q_9 & q_2 & q_5 \in F
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Since the sequence of states is of length $|s| + 1 > \ell$, and there are only $\ell$ different states in $Q$, at least one state is repeated (by the pigeonhole principle).
Write down $s = xyz$

By inspection, $M$ accepts $xy^kz$ for every $k \geq 0$. 
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$|y| > 0$ because the state ($q_9$ in figure) is repeated.
Pumping Lemma – Proof (cont.)

Write down $s = xyz$

By inspection, $M$ accepts $xy^kz$ for every $k \geq 0$.

$|y| > 0$ because the state ($q_9$ in figure) is repeated.

To ensure that $|xy| \leq \ell$, pick first state repetition, which must occur no later than $\ell + 1$ states in sequence.
Theorem: The language $B = \{0^n1^n | n > 0\}$ is not regular.

Proof: By contradiction. Suppose $B$ is regular, accepted by DFA $M$. Let $\ell$ be the pumping length.

Consider the string $s = 0^\ell 1^\ell$. 
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Proof: By contradiction. Suppose $B$ is regular, accepted by DFA $M$. Let $\ell$ be the pumping length.

Consider the string $s = 0^\ell 1^\ell$.

By pumping lemma $s = xyz$, where $xy^kz \in B$ for every $k$. 
An Application

Theorem: The language \( B = \{0^n1^n|n > 0\} \) is not regular.

Proof: By contradiction. Suppose \( B \) is regular, accepted by DFA \( M \). Let \( \ell \) be the pumping length.

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- By pumping lemma \( s = xyz \), where \( xy^kz \in B \) for every \( k \).
- If \( y \) is all 0, then \( xy^kz \) has too many 0’s.
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If \( y \) is all 0, then \( xy^kz \) has too many 0’s.

If \( y \) is all 1, then \( xy^kz \) has too many 1’s.

If \( y \) is mixed, then \( xy^kz \) is not of right form.

♣
Another Application

**Theorem:** The language $C = \{w|w \text{ has an equal number of 0's and 1's}\}$ is not regular.

**Proof:** By contradiction. Suppose $C$ is regular, accepted by DFA $M$. Let $\ell$ be the pumping length.

- Consider the string $s = 0^\ell 1^\ell$. 
Another Application

Theorem: The language $C = \{w|w\text{ has an equal number of 0's and 1's}\}$ is not regular.

Proof: By contradiction. Suppose $C$ is regular, accepted by DFA $M$. Let $\ell$ be the pumping length.

- Consider the string $s = 0^\ell 1^\ell$.
- By pumping lemma $s = xyz$, where $xy^kz \in C$ for every $k$. 
Another Application

**Theorem:** The language $C = \{ w | w \text{ has an equal number of 0's and 1's} \}$ is not regular.

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- Consider the string $s = 0^\ell 1^\ell$.
- By pumping lemma $s = xyz$, where $xy^kz \in C$ for every $k$.
- If $y$ is all 0, then $xy^kz$ has too many 0's.
Another Application

**Theorem:** The language 

\[ C = \{ w \mid w \text{ has an equal number of 0's and 1's} \} \]

is not regular.

**Proof:** By contradiction. Suppose \( C \) is regular, accepted by DFA \( M \). Let \( \ell \) be the pumping length.

- Consider the string \( s = 0^\ell 1^\ell \).
- By pumping lemma \( s = xyz \), where \( xy^kz \in C \) for every \( k \).
- If \( y \) is all 0, then \( xy^kz \) has too many 0’s.
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- If $y$ is all 0, then $xy^kz$ has too many 0’s.
- If $y$ is all 1, then $xy^kz$ has too many 1’s.
- If $y$ is mixed, then since $|xy| \leq \ell$, $y$ must be all 0’s, contradiction.

♣
Algorithms for NDA’s

Given an NDA, \( N \), and a string \( s \), is \( s \in L(N) \)?

**Answer:** Construct the DFA equivalent to \( N \) and run it on \( w \).
Given an NDA, $N$, and a string $s$, is $s \in L(N)$?

**Answer:** Construct the DFA equivalent to $N$ and run it on $w$.

Is $L(N) = \emptyset$?

**Answer:** This is a reachability question in graphs: Is there a path in the states’ graph of $N$ from the start state to some accepting state. There are simple, efficient algorithms for this task.
More Algorithms for NDA’s

Is $L(N) = \Sigma^*$?

Answer: Check if $\overline{L(N)} = \emptyset$. 

Given $N_1$ and $N_2$, is $L(N_1) \subseteq L(N_2)$?

Answer: Check if $L(N_2) \cap L(N_1) = \emptyset$. 

Given $N_1$ and $N_2$, is $L(N_1) = L(N_2)$?

Answer: Check if $L(N_1) \subseteq L(N_2)$ and $L(N_2) \subseteq L(N_1)$. 

Slides modified by Benny Chor, based on original slides by Maurice Herlihy, Brown University.
More Algorithms for NDA’s

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Given \( N_1 \) and \( N_2 \), is \( L(N_1) = L(N_2) \)?

**Answer:** Check if \( L(N_1) \subseteq L(N_2) \) and \( L(N_2) \subseteq L(N_1) \).