DFA Formal Definition (reminder)

A deterministic finite automaton (DFA) is a 5-tuple 
\((Q, \Sigma, \delta, q_0, F)\), where

- \(Q\) is a finite set called the states,
- \(\Sigma\) is a finite set called the alphabet,
- \(\delta : Q \times \Sigma \rightarrow Q\) is the transition function,
- \(q_0 \in Q\) is the start state, and
- \(F \subseteq Q\) is the set of accept states.
Back to $M_1$

$M_1 = (Q, \Sigma, \delta, q_1, F)$ where

- $Q = \{q_1, q_2, q_3\}$,
- $\Sigma = \{0, 1\}$,
Back to $M_1$

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Slides modified by Benny Chor, based on original slides by Maurice Herlihy, Brown University.
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  $\begin{array}{c|cc}
    & 0 & 1 \\
  \hline
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  \end{array}$

- $q_1$ is the start state, and $F = \{q_2\}$.  

Back to $M_1$
Another Example
And Yet Another Example

\[ q_0 \xrightarrow{0} q_1 \xrightarrow{2, \text{RESET}} q_0 \]
\[ q_0 \xrightarrow{0} q_2 \xrightarrow{0, \text{RESET}} q_0 \]
\[ q_1 \xrightarrow{1} q_2 \xrightarrow{1} q_1 \]
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A Formal Model of Computation

- Let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA, and
- let $w = w_1w_2 \cdots w_n$ be a string over $\Sigma$.

We say that $M$ accepts $w$ if there is a sequence of states $r_0, \ldots, r_n$ ($r_i \in Q$) such that
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- $r_n \in F$
The Regular Operations

Let $A$ and $B$ be languages.

The union operation:

$$A \cup B = \{ x | x \in A \text{ or } x \in B \}$$

The concatenation operation:

$$A \circ B = \{ xy | x \in A \text{ and } y \in B \}$$

The star operation:

$$A^* = \{ x_1 x_2 \ldots x_k | k \geq 0 \text{ and each } x_i \in A \}$$
The Regular Operations – Examples

Let $A = \{\text{good, bad}\}$ and $B = \{\text{boy, girl}\}$.

Union

\[ A \cup B = \{\text{good, bad, boy, girl}\} \]

Concatenation

\[ A \circ B = \{\text{goodboy, goodgirl, badboy, badgirl}\} \]

Star

\[ A^* = \{\varepsilon, \text{good, bad, goodgood, goodbad, badbad, badgood, \ldots}\} \]
Claim: Closure Under Union

If $A_1$ and $A_2$ are regular languages, so is $A_1 \cup A_2$. 

Approach to Proof:

Some $M_1$ accepts $A_1$ and $M_2$ accepts $A_2$. Construct a new machine $M$ that accepts $A_1 \cup A_2$.

Attempted Proof Idea:

First simulate $M_1$ and if $M_1$ doesn't accept, then simulate $M_2$.

What's wrong with this?

Fix: Simulate both machines simultaneously.
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Closure Under Union: Correct Proof

Suppose $M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ accepts $L_1$,
and $M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ accepts $L_2$.
Closure Under Union: Correct Proof

Suppose $M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ accepts $L_1$, and $M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ accepts $L_2$.

Define $M$ as follows ($M$ will accept $L_1 \cup L_2$):

- $Q = Q_1 \times Q_2$.
- $\Sigma$ is the same.
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- \( q_0 = (q_1, q_2) \)
- \( F = \{ (r_1, r_2) | r_1 \in F_1 \text{ or } r_2 \in F_2 \} \).

(hey, why not choose \( F = F_1 \times F_2 \)?)
What About Concatenation?

**Thm:** If $L_1$, $L_2$ are regular languages, so is $L_1 \circ L_2$.

**Example:** $L_1 = \{\text{good, bad}\}$ and $L_2 = \{\text{boy, girl}\}$.

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This is much harder to prove.

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**Idea:** Simulate $M_1$ for a while, then switch to $M_2$.

**Problem:** But when do you switch?

This leads us into non-determinism.
Non-Deterministic Finite Automata

- an NFA may have **more than one transition labeled with a certain symbol**, 

![Diagram of NFA with labels and transitions](image)
Non-Deterministic Finite Automata

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Slide modified by Benny Chor, based on original slides by Maurice Herlihy, Brown University.
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- transitions may be labeled with $\varepsilon$, the empty string.

Comment: Every DFA is also a non-deterministic finite automata (NFA).
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Non-Deterministic Computation

What happens when more than one transition is possible?

What does an $\varepsilon$ transition do?
Non-Deterministic Computation

What happens when more than one transition is possible?

- The machine “splits” into multiple copies
- Each branch follows one possibility
- Together, branches follow all possibilities.
- If the input doesn’t appear, that branch “dies”.
- Automaton accepts if some branch accepts.
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Non-Deterministic Computation

What happens on string 1001?
The String 1001
The String 1001

symbol
1

0
0
0
1
1

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Why Non-Determinism?

Theorem: Deterministic and non-deterministic finite automata accept exactly the same set of languages.
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Q.: So why do we need them?
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**Theorem:** Deterministic and non-deterministic finite automata accept exactly the same set of languages.

**Q.:** So why do we need them?

**A.:** NFAs are usually easier to design than equivalent DFAs.
Why Non-Determinism?

**Theorem:** Deterministic and non-deterministic finite automata accept exactly the same set of languages.

Q.: So why do we need them?

A.: NFAs are usually easier to design than equivalent DFAs.

**Example:** Design a finite automaton that accepts all strings with a 1 in their third-to-the-last position?
A Deterministic Automaton

(there are a few errors, e.g. $q_{101}$ should be an accept state, but overall it is OK.)
A Non-Deterministic Automaton

- "Guesses" which symbol is third from the last, and
- checks that it’s a 1.
NFA – Formal Definition

Transition function $\delta$ is going to be different.

- $\mathcal{P}(Q)$ is the powerset of $Q$.
- $\Sigma_\varepsilon = \Sigma \cup \{\varepsilon\}$. 
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- $\mathcal{P}(Q)$ is the powerset of $Q$.
- $\Sigma_\varepsilon = \Sigma \cup \{\varepsilon\}$.

A **non-deterministic** finite automaton is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

- $Q$ is a finite set called the **states**, 
- $\Sigma$ is a finite set called the **alphabet**, 
- $\delta : Q \times \Sigma_\varepsilon \rightarrow \mathcal{P}(Q)$ is the **transition function**, 
- $q_0 \in Q$ is the **start state**, and 
- $F \subseteq Q$ is the set of **accept states**.
Example

\[ N_1 = (Q, \Sigma, \delta, q_1, F) \text{ where } \]

\[ Q = \{q_1, q_2, q_3, q_4\}, \quad \Sigma = \{0, 1\}, \]
Example

\[ N_1 = (Q, \Sigma, \delta, q_1, F) \] where

- \( Q = \{q_1, q_2, q_3, q_4\} \), \( \Sigma = \{0, 1\} \),

\[ \begin{array}{c|ccc}
  & 0 & 1 & \varepsilon \\
\hline
q_1 & \{q_1, q_2\} & \{q_1\} & \emptyset \\
q_2 & \{q_3\} & \emptyset & \{q_3\} \\
q_3 & \emptyset & \{q_4\} & \emptyset \\
q_4 & \{q_4\} & \{q_4\} & \emptyset \\
\end{array} \]

\( q_1 \) is the start state, and \( F = \{q_4\} \).
Example

\[ N_1 = (Q, \Sigma, \delta, q_1, F) \] where

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</tr>
<tr>
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Formal Model of Computation

Let $M = (Q, \Sigma, \delta, q_0, F)$ be an NFA, and

$w$ be a string over $\Sigma_\varepsilon$ that has the form $y_1 y_2 \cdots y_m$ where $y_i \in \Sigma_\varepsilon$.

$u$ be the string over $\Sigma$ obtained from $w$ by omitting all occurrences of $\varepsilon$. 
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Suppose there is a sequence of states (in $Q$), $r_0, \ldots, r_n$, such that

- $r_0 = q_0$
- $\delta(r_i, y_{i+1}) \in r_{i+1}$, $0 \leq i < n$
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Formal Model of Computation

- Let \( M = (Q, \Sigma, \delta, q_0, F) \) be an NFA, and
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- \( u \) be the string over \( \Sigma \) obtained from \( w \) by omitting all occurrences of \( \varepsilon \).

Suppose there is a sequence of states (in \( Q \)), \( r_0, \ldots, r_n \), such that

- \( r_0 = q_0 \)
- \( \delta(r_i, y_{i+1}) \in r_{i+1}, 0 \leq i < n \)
- \( r_n \in F \)

Then we say that \( M \) accepts \( u \).
Equivalence of NFA’s and DFA’s

Given an an NFA, $N$ then we construct a DFA, $M$, that accepts the same language.
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- Make DFA simulate all possible NFA states.
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- Given an an NFA, $N$ then we construct a DFA, $M$, that accepts the same language.
- To begin with, we make things easier by ignoring $\varepsilon$ transitions.
- Make DFA simulate all possible NFA states.
- As consequence of the construction, if the NFA has $k$ states, the DFA has $2^k$ states.
Equivalence of NFA’s and DFA’s

Let $N = (Q, \Sigma, \delta, q_0, F)$ be the NFA accepting $A$. Construct a DFA $M = (Q', \Sigma, \delta', q'_0, F')$.

- $Q' = \mathcal{P}(Q)$. 
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Construct a DFA \( M = (Q', \Sigma, \delta', q'_0, F') \).

- \( Q' = \mathcal{P}(Q) \).
- For \( R \in Q' \) and \( a \in \Sigma \), let

\[
\delta'(R, a) = \{ q \in Q | q \in \delta(r, a) \text{ for some } r \in R \}
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- \( q'_0 = \{q_0\} \)
- \( F' = \{ R \in Q' \mid R \text{ contains an accept state of } N \} \)
Dealing with $\varepsilon$-Transitions

For any state $R$ of $M$, define $E(R)$ to be the collection of states reachable from $R$ by $\varepsilon$ transitions only.

$$E(R) = \{ q \in Q | q \text{ can be reached from some } r \in R \text{ by 0 or more } \varepsilon \text{ transitions} \}$$
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Define transition function:

$$\delta'(R, a) = \{ q \in Q | \text{ there is some } r \in R \text{ such that } q \in E(\delta(r, a)) \}$$
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Change start state to

$$q'_0 = E(\{q_0\})$$
Regular Languages, Revisited

By definition, a language is regular if it is accepted by some DFA.
Regular Languages, Revisited

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\textbf{Corollary:} A language is regular if and only if it is accepted by some \textbf{NFA}.
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This is an alternative way of characterizing regular languages.
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This is an alternative way of characterizing regular languages.

We will now use the equivalence to show that regular languages are closed under the regular operations (union, concatenation, star).
Regular Languages Closed Under Union

N_1

N_2
Regular Languages Closed Under Union

Slides modified by Benny Chor, based on original slides by Maurice Herlihy, Brown University.
Regular Languages Closed Under Union

Suppose

- $N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ accept $L_1$, and
- $N_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ accept $L_2$.

Define $N = (Q, \Sigma, \delta, q_0, F)$:

- $Q = q_0 \cup Q_1 \cup Q_2$
- $\Sigma$ is the same, $q_0$ is the start state
- $F = F_1 \cup F_2$

$$\delta'(q, a) = \begin{cases} 
\delta_1(q, a) & q \in Q_1 \\
\delta_2(q, a) & q \in Q_2 \\
\{q_1, q_2\} & q = q_0 \text{ and } a = \varepsilon \\
\emptyset & q = q_0 \text{ and } a \neq \varepsilon 
\end{cases}$$
Regular Languages Closed Under Concatenation

\[ N_2 \]

\[ N_1 \]

Slides modified by Benny Chor, based on original slides by Maurice Herlihy, Brown University.
Regular Languages
Closed Under Concatenation

\[ \varepsilon \]

\[ N_1 \]

\[ N_2 \]

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Regular Languages
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Regular Languages
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Define $N = (Q, \Sigma, \delta, q_1, F_2)$:
- $Q = Q_1 \cup Q_2$
- $q_1$ is the start state of $N$
Regular Languages
Closed Under Concatenation

Suppose

- \( N_1 = (Q_1, \Sigma, \delta_1, q_1, F_1) \) accept \( L_1 \), and
- \( N_2 = (Q_2, \Sigma, \delta_2, q_2, F_2) \) accept \( L_2 \).

Define \( N = (Q, \Sigma, \delta, q_1, F_2) \):

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- \( q_1 \) is the start state of \( N \)
- \( F_2 \) is the set of accept states of \( N \)

\[
\delta'(q, a) = \begin{cases} 
\delta_1(q, a) & q \in Q_1 \text{ and } q \notin F \\
\delta_1(q, a) & q \in Q_1 \text{ and } a \neq \varepsilon \\
\delta_1(q, a) \cup \{q_2\} & q \in F_1 \text{ and } a = \varepsilon \\
\delta_2(q, a) & q \in Q_2 
\end{cases}
\]
Regular Languages Closed Under \textit{Star}

\textbf{N}_1

Oops - bad construction. How do we fix it?
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Summary

- Regular languages are closed under

Slides modified by Benny Chor, based on original slides by Maurice Herlihy, Brown University.
Summary

- Regular languages are closed under
  - union
Summary

- Regular languages are closed under
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Non-deterministic finite automata are equivalent to deterministic finite automata but much easier to use in some proofs and constructions.
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Regular Expressions

A notation for building up languages by describing them as expressions, \( e.g. (0 \cup 1)0^\ast \).

- \( 0 \) and \( 1 \) are shorthand for \( \{0\} \) and \( \{1\} \)
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Q.: What does \((0 \cup 1)0^*\) stand for?
Regular Expressions

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**Remark:** Regular expressions are often used in text editors or shell scripts.
More Examples

Let $\Sigma$ be an alphabet.

- The regular expression $\Sigma$ is the language of one-symbol strings.
- $\Sigma^*$ is all strings.
- $\Sigma^*1$ all strings ending in 1.
- $0\Sigma^* \cup \Sigma^*1$ strings starting with 0 or ending in 1.
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Just like in arithmetic, operations have precedence:

- star first
- concatenation next
- union last
- parentheses used to change usual order
Regular Expressions – Formal Definition

Syntax: \( R \) is a regular expression if \( R \) is of form

- \( a \) for some \( a \in \Sigma \)
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Let $L(R)$ be the language denoted by regular expression $R$.

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**Thm.** A language, $L$, is described by a regular expression, $R$, if and only if $L$ is regular.
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$\impliedby$ Given a regular language, $L$, construct an equivalent regular expression.