The Price of Anarchy and the Design of Scalable Resource Allocation Mechanisms

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Algorithmic Game Theory Seminar - TAU

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Resource allocation

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- No algorithms described, but communication overhead should be low (users choose a single parameter).
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- We would like to characterize the efficiency of a general class of mechanisms.
- No algorithms described, but communication overhead should be low (users choose a single parameter).
- We would investigate a VCG approach.
Problem Definition

- Notation:

\[\begin{align*}
R & \text{users} \\
\mathbf{C} & \text{a resource of capacity} \quad \mathbf{C} > 0 \text{ which they share} \\
d_r & \text{the amount of} \quad \mathbf{C} \text{ allocated to user} \\
U_r(d_r) & \text{the utility user} \quad r \text{ gets} \\
\end{align*}\]

Goal (denoted from now on as \(\text{SYSTEM}\)):

\[
\max \sum_r U_r(d_r) \quad \text{subject to} \quad \sum_r d_r \leq \mathbf{C} \quad d_r \geq 0 \quad \forall r \in R
\]
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$$\text{maximize } \sum_r U_r(d_r)$$

subject to

$$\sum_r d_r \leq C$$

$$d_r \geq 0 \quad \forall r \in R$$
Formalities

- **Assumption 1:**

\[ \text{for each } r, \text{ over the domain } d_r \geq 0 \text{ the utility function } U_r(d_r) \text{ is concave, strictly increasing and continuous} \]

\[ \text{over the domain } d_r > 0, U_r(d_r) \text{ is continuously differentiable} \]

\[ \text{the right directional derivative at 0 (denoted } U'_{r}(0) \text{) is finite} \]

\[ U^- \text{ the set of all utility functions satisfying these conditions} \]

- the objective function is the utilitarian social welfare function
- the objective function is continuous and the feasible region is compact
- an optimal solution \( d = (d_1, \ldots, d_R) \) exists
- \( U_r \) are strictly concave - the optimal solution is unique (feasible region is convex)

- obvious problem
- \( U_r \text{ aren't available to the resource manager (strategic players)} \)
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- must satisfy \(\sum_r w_r / \mu = C\) (eating the whole cake)
- we get \(\mu = \frac{\sum_r w_r}{C}\)
Competitive Equilibrium for *price takers*

- users are *price takers* - each user does not anticipate the effect of their payment $w_r$ on the price $\mu$

\[ P_r(w_r; \mu) = U_r(w_r, \mu) - w_r \] over $w_r \geq 0$

similar to value

A pair $((w_1, \ldots, w_R), \mu)$ is a competitive equilibrium if for every user $r$:

\[ P_r(w_r; \mu) \geq P_r(w_r; \mu) \quad \forall w_r \geq 0 \]

$\mu = \sum_r w_r$

intuitively - each user gets the best value with its current payment and the “whole pie” is eaten
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- given a price $\mu > 0$, user $r$ wishes to maximize the following:

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**Theorem**

*There exists a competitive equilibrium \((w, \mu)\), and for which, the vector \(d = w/\mu\) is an optimal solution for SYSTEM.*
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**Proof.**

- The payoff function \((P_r(w_r; \mu))\) is concave for any \(\mu > 0\).
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- If \(d_r \geq 0\) then \(P_r(w_r; \mu)' = \frac{1}{\mu} U'_r(d_r) - 1 = 0 \Rightarrow U'_r(d_r) = \mu\).
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- If \(d_r = 0\) then \(U'_r(0) \leq \mu\).
- From the second equilibrium condition we get that \(C = \sum_r \frac{w_r}{\mu} = \sum_r d_r\).
Competitive Equilibrium for price takers

proof-cont.

- Using simple Lagrangian optimization, we get that these conditions are exactly the optimality conditions for $SYSTEM$. 
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Therefore, \((w, \mu)\) is a competitive equilibrium \(\iff d = w/\mu\) is a solution to SYSTEM with Lagrange multiplier \(\mu\).
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proof-cont.

- Using simple Lagrangian optimization, we get that these conditions are exactly the optimality conditions for \( \text{SYSTEM} \).
- Therefore, \((w, \mu)\) is a competitive equilibrium \( \iff d = w/\mu \) is a solution to \( \text{SYSTEM} \) with Lagrange multiplier \( \mu \).
- There exists a unique solution to \( \text{SYSTEM} \) \( \Rightarrow \) there exists a competitive equilibrium.
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Meaning - if users are \emph{price takers} then there exists an equilibrium \( w \) where all users have optimally chosen their bids \( w_r \) with respect to the given price \( \mu = \frac{\sum_r w_r}{c} \) and the aggregate utility is maximized.
Nash Equilibrium for *price anticipators*

- Users are smart - they realize that $\mu = \frac{\sum_r w_r}{C}$.
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- $w_{-r} = (w_1, w_2, \ldots, w_{r-1}, w_{r+1}, \ldots, w_R)$ (everybody except me)
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- given $w_{-r}$, user $r$ chooses $w_r$ to maximize:

$$Q_r(w_r; w_{-r}) = \begin{cases} U_r \left( \frac{w_r}{\sum_{s} w_s} C \right) - w_r & w_r > 0 \\ U_r(0) & w_r = 0 \end{cases}$$
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- A *Nash equilibrium* is a bid $w$ such that for all $r$:

$$Q_r(w_r; w_{-r}) \geq Q_r(\overline{w}_r; w_{-r}) \quad \forall \overline{w}_r \geq 0$$
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**Theorem**

*Suppose* $R > 1$. *Then there exists a unique Nash equilibrium* $\mathbf{w} \geq 0$ *of the game defined by* $(Q_1, \ldots, Q_r)$, *and it satisfies* $\sum_r w_r > 0$. 

Nash Equilibrium for price anticipators

Theorem

- Suppose $R > 1$. Then there exists a unique Nash equilibrium $w \geq 0$ of the game defined by $(Q_1, ..., Q_r)$, and it satisfies $\sum_r w_r > 0$.

- The vector $d$ defined by $d_r = \frac{w_r}{\sum_s w_s} C$ is a unique optimal solution for the following optimization problem (denoted as GAME):

  \[
  \text{maximize } \sum_r \hat{U}_r(d_r) \\
  \text{subject to } \sum_r d_r \leq C \\
  d_r \geq 0 \quad \forall r \in [R]
  \]

  where $\hat{U}_r(d_r) = \left(1 - \frac{d_r}{C}\right) U_r(d_r) + \frac{d_r}{C} \left(\frac{1}{d_r} \int_0^{d_r} U_r(z)dz\right)$.
Nash Equilibrium for *price anticipators*

**Proof.**

- At any Nash equilibrium, at least two bidders with positive bid (otherwise, the only bidder can always offer less and increase his gain) - therefore the payoff function is concave and continuous in $w_r$. 
Nash Equilibrium for *price anticipators*

**Proof.**

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- The same as with the previous theorem, we can find the conditions for maxima:

$$U'_r \left( \frac{w_r}{\sum_s w_s} C \right) \left( 1 - \frac{w_r}{\sum_s w_s} \right) = \frac{\sum_s w_s}{C} \quad w_r > 0$$

$$U'_r(0) \leq \frac{\sum_s w_s}{C} \quad w_r = 0$$
Nash Equilibrium for *price anticipators*

**proof-cont.**

Let $\rho = \sum_s w_s / C$ and $d_r = w_r / \rho$, we can rewrite the previous:

\[
\hat{U}_r'(d_r) = \rho \quad d_r > 0 \\
\hat{U}_r'(0) \leq \rho \quad d_r = 0 \\
\sum_r d_r = C
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Nash Equilibrium for price anticipators

proof-cont.

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- We get the same optimality conditions as we got with competitive equilibrium, but under a different objective function.
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- We get the same optimality conditions as we got with competitive equilibrium, but under a different objective function.

- Since the utility functions \( \hat{U}_r(d_r) \) are strictly concave and continuous over \( 0 \leq d_r \leq C \), these conditions are sufficient optimality for \( GAME \).
Nash Equilibrium for *price anticipators*

proof-cont.

Hence, \( \mathbf{w} \) is a Nash equilibrium if and only if \( \sum_s w_s > 0 \) and the resulting allocation \( \mathbf{d} \) is optimal for \( \text{GAME} \) with Lagrange multiplier \( \rho = \sum_s w_s / C \).
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proof-cont.

- Hence, \( \mathbf{w} \) is a Nash equilibrium \( \iff \sum_s w_s > 0 \) and the resulting allocation \( \mathbf{d} \) is optimal for GAME with Lagrange multiplier \( \rho = \sum_s w_s / C \).
- GAME has a strictly concave and continuous objective function over a compact feasible region \( \implies \) unique optimal solution \( \implies \) unique Nash equilibrium.
Nash Equilibrium for \textit{price anticipators}

\begin{itemize}
  \item Hence, \(\mathbf{w}\) is a Nash equilibrium \(\iff \sum_s w_s > 0\) and the resulting allocation \(\mathbf{d}\) is optimal for \(\text{GAME}\) with Lagrange multiplier \(\rho = \sum_s w_s / C\).
  \item \(\text{GAME}\) has a strictly concave and continuous objective function over a compact feasible region \(\Rightarrow\) unique optimal solution \(\Rightarrow\) unique Nash equilibrium.
\end{itemize}
Corollary of the Theorem

**Corollary**

Suppose $R > 1$. Let $\mathbf{w}$ be the unique Nash equilibrium of the game defined by $(Q_1, \ldots, Q_R)$, and $\mathbf{d}$ defined by $d_r = \frac{w_r}{\sum_s w_s} C$, then for every $\mathbf{d} \geq 0$ such that $\sum_r d_r \leq C$, there holds:

$$\sum_r \hat{U}'_r(d_r)(\bar{d}_r - d_r) \leq 0$$
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Suppose $R > 1$. Let $w$ be the unique Nash equilibrium of the game defined by $(Q_1, \ldots, Q_R)$, and $d$ defined by $d_r = \frac{w_r}{\sum_s w_s} C$, then for every $\overline{d} \geq 0$ such that $\sum_r d_r \leq C$, there holds:

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Proof of the Corollary

Proof.

\[ \sum_r \hat{U}_r'(d_r)(\overline{d}_r - d_r) \leq \]

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Proof of the Corollary

Proof.

\[ \sum_r \hat{U}_r'(d_r)(\bar{d}_r - d_r) \leq \sum_r \frac{\sum_s w_s}{C} (\bar{d}_r - \frac{w_r}{\sum_s w_s} C) = \]
Proof of the Corollary

Proof.

\[ \sum_r \hat{U}_r'(d_r)(\bar{d}_r - d_r) \leq \sum_r \frac{\sum_s w_s}{C} (\bar{d}_r - \frac{w_r}{\sum_s w_s} C) = \]

\[ \sum_r \bar{d}_r \frac{\sum_s w_s}{C} - \sum_r w_r \leq \]
Proof of the Corollary

Proof.

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\[ \sum_r \bar{d}_r \frac{\sum_s w_s}{C} - \sum_r w_r \leq C \frac{\sum_s w_s}{C} - \sum_r w = \]
Proof of the Corollary

Proof.

\[ \sum_r \hat{U}_r'(d_r)(\bar{d}_r - d_r) \leq \sum_r \frac{\sum_s w_s}{C} (\bar{d}_r - \frac{w_r}{\sum_s w_s} C) = \]

\[ \sum_r \frac{w_r}{C} \sum_s w_s - \sum_r w_r \leq C \frac{\sum_s w_s}{C} - \sum_r w = 0 \]
Measuring Anarchy

- Let’s denote the optimal solution for SYSTEM as $d^S$ and $d^G$ as the optimal solution for GAME.
Measuring Anarchy

- Let’s denote the optimal solution for SYSTEM as \(d^S\) and \(d^G\) as the optimal solution for GAME.
- Price of anarchy - how much utility is lost because users are price anticipating?
Measuring Anarchy

- Let’s denote the optimal solution for $SYSTEM$ as $d^S$ and $d^G$ as the optimal solution for $GAME$.
- Price of anarchy - how much utility is lost because users are *price anticipating*?
- We know that $\sum_r U_r(d^G) \leq \sum_r U_r(d^S)$, but by how much?
Theorem

Suppose that $R > 1$ and that $U_r(0) \geq 0$ for all $r$. We can bound the efficiency loss by the following:

$$\sum_r U_r(d_r^G) \geq \frac{3}{4} \sum_r U_r(d_r^S)$$

Furthermore, this bound is tight, i.e., for every $\epsilon > 0$, exists $R$ and utility functions $U_r$ for every user $r$ such that:

$$\sum_r U_r(d_r^G) \leq \left( \frac{3}{4} + \epsilon \right) \sum_r U_r(d_r^S)$$
Proof of the Bound

Proof.

\[ \beta = \inf_{U \in \mathcal{U}} \inf_{c > 0} \inf_{0 \leq d, \bar{d} \leq c} \frac{U(d) + \hat{U}'(d)(\bar{d} - d)}{U(d)} \]
Proof of the Bound

Proof.

- \( \beta = \inf_{u \in U} \inf_{c > 0} \inf_{0 \leq d, \bar{d} \leq c} \frac{U(d) + \hat{U}'(d)(\bar{d} - d)}{U(d)} \)

- \( d \geq \bar{d} \Rightarrow \beta \geq 1 \) (\( \beta \) is increasing in \( d \) and \( \beta = 1 \) when \( d = \bar{d} \)).
Proof of the Bound

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- If \( d < \bar{d} \), then:
Proof of the Bound

Proof.

1. \( \beta = \inf_{U \in \mathcal{U}} \inf_{C > 0} \inf_{0 \leq d, \bar{d} \leq C} \frac{U(d) + \hat{U}'(d)(\bar{d} - d)}{U(d)} \)
2. \( d \geq \bar{d} \Rightarrow \beta \geq 1 \) (\( \beta \) is increasing in \( d \) and \( \beta = 1 \) when \( d = \bar{d} \)).
3. If \( d < \bar{d} \), then:

\[
U(d) + \hat{U}'(d)(\bar{d} - d) = \]

\[
\geq \frac{3}{4} U(d) + (1 - d) (U(d) - U(d))
\]
Proof of the Bound

Proof.

- \( \beta = \inf_{u \in U} \inf_{c > 0} \inf_{0 \leq d, \bar{d} \leq c} \frac{U(d) + \hat{U}'(d)(\bar{d} - d)}{U(d)} \)
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- If \( d < \bar{d} \), then:

\[
U(d) + \hat{U}'(d)(\bar{d} - d) = U(d) + U'(d) \left(1 - \frac{d}{C}\right) (\bar{d} - d)
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Proof of the Bound

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- If \( d < \bar{d} \), then:

\[
U(d) + \hat{U}'(d)(\bar{d} - d) = U(d) + U'(d) \left(1 - \frac{d}{\bar{d}}\right)(\bar{d} - d) \\
\geq U(d) + \left(1 - \frac{d}{\bar{d}}\right)(U(\bar{d}) - U(d))
\]
Proof of the Bound

**Proof.**

- \( \beta = \inf_{U \in \mathcal{U}} \inf_{C > 0} \inf_{0 \leq d, \overline{d} \leq C} \frac{U(d) + \hat{U}'(d)(\overline{d} - d)}{U(d)} \)

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\[
\geq U(d) + \left(1 - \frac{d}{\overline{d}}\right)(U(\overline{d}) - U(d))
\]

\[
\geq \left(\frac{d}{\overline{d}}\right)^2 U(\overline{d}) + \left(1 - \frac{d}{\overline{d}}\right) U(\overline{d})
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Proof of the Bound

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\[ \beta = \inf_{u \in U} \inf_{c > 0} \inf_{0 \leq d, \bar{d} \leq c} \frac{U(d) + \hat{U}'(d)(\bar{d} - d)}{U(d)} \]

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\geq \left( \frac{d}{\bar{d}} \right)^2 U(\bar{d}) + \left( 1 - \frac{d}{\bar{d}} \right) U(\bar{d}) \geq \frac{3}{4} U(\bar{d})
\]
Proof of the Bound

Proof-cont.

Let \( d^S \) and \( d^G \) be the solutions to SYSTEM and GAME.
Proof of the Bound

Let $d^S$ and $d^G$ be the solutions to SYSTEM and GAME.

Since:

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\sum_s U_s(d^S_s) \leq \sum_s \frac{1}{\beta} \left( U_s(d^G_s) + \hat{U}'_s(d^G_s)(d^S_s - d^G_s) \right) \leq \frac{1}{\beta} \sum_s U_s(d^G_s)
$$

we get a bound on the anarchy of $\frac{4}{3}$. 
Proof of the Bound

Proof-cont.

- Let \( d^S \) and \( d^G \) be the solutions to \( \text{SYSTEM} \) and \( \text{GAME} \).
- Since:

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we get a bound on the anarchy of \( \frac{4}{3} \).

- Next step - show the bound is tight.
Proof of the Bound Tightness

Proof-cont.

- Fix $U, d < \bar{d}$ and set $C = \bar{d}$. Suppose $R > 1$ users.
Proof of the Bound Tightness

Proof-cont.

- Fix $U, d < \overline{d}$ and set $C = \overline{d}$. Suppose $R > 1$ users.
- Let $U_1 = U$ and $U_r(d_r) = \hat{U}'(d)d_r = (U'(d)(1 - d/C))d_r$ for $r = 2, ..., R$. 

Since a possible solution involves giving all the resource to the first user, it's obvious that $\sum_{s} U_s(d_S) \geq U(d)$. Nash equilibrium has at least 2 users with a positive quantity and is unique $\Rightarrow$ users $2, ..., R$ receive the same quantity.

$\lim_{R \to \infty} d_r = 0$ for $r = 2, ..., R$. 

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Proof of the Bound Tightness

Proof-cont.

- Fix $U$, $d < \bar{d}$ and set $C = \bar{d}$. Suppose $R > 1$ users.
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\[\text{Nash equilibrium has at least 2 users with a positive quantity and is unique} \Rightarrow \text{users 2, ..., R receive the same quantity.}\]

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Proof of the Bound Tightness

Proof-cont.

- Fix \( U, d < \bar{d} \) and set \( C = \bar{d} \). Suppose \( R > 1 \) users.
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Proof-cont.

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- $\lim_{R \to \infty} d_r = 0$ for $r = 2, \ldots, R$. 
Proof of the Bound Tightness

Proof-cont.

Out of the equilibrium conditions, we get that when $R \to \infty$:

$$
\sum_s \frac{w_s}{C} = \hat{U}_r'(d_r) = (1 - \frac{d_r}{C})U_r'(d_r) \approx \hat{U}'(d)
$$
Proof of the Bound Tightness

Proof-cont.

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\]

- Hence, at a Nash equilibrium, \( d_1 = d + \epsilon \) and \( d_r = \frac{d - d - \epsilon}{R - 1} \) for \( r = 2, \ldots, R \) where \( \lim_{R \to \infty} \epsilon = 0 \).
Proof of the Bound Tightness

Proof-cont.

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- Thus, we get an efficiency of
  \[
  \frac{\sum_s U_s(d^G_s)}{\sum_s U_s(d^S_s)} \ll \frac{U(d) + \hat{U}'(d)(\bar{d} - d)}{U(d)}
  \]
Proof of the Bound Tightness

Proof-cont.

- Out of the equilibrium conditions, we get that when $R \to \infty$:

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$$\frac{\sum_s U_s(d_s^G)}{\sum_s U_s(d_s^S)} \leq \frac{U(d) + \hat{U}'(d)(\bar{d} - d)}{U(d)}.$$

- For $d = \frac{1}{2}$, $\bar{d} = 1$ and $U(x) = x$ we get that

$$\frac{\sum_s U_s(d_s^G)}{\sum_s U_s(d_s^S)} \leq \frac{3}{4}.$$
Proof of the Bound Tightness

Proof-cont.

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  \]

- Meaning - the price of anarchy is exactly $\frac{4}{3}$. 

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Proof of the Bound Tightness

Proof-cont.

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- Hence, at a Nash equilibrium, $d_1 = d + \epsilon$ and $d_r = \frac{d-d-\epsilon}{R-1}$ for $r = 2, \ldots, R$ where $\lim_{R \to \infty} \epsilon = 0$.

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- Meaning - the price of anarchy is exactly $\frac{4}{3}$. 

The Price of Anarchy and the Design of Scalable Resource Allocation Mechanisms

Alon Ardenboim
We want to find the conditions for a mechanism so it would be “Desirable” and “Reasonable”.
The Search for “Desirable” and “Reasonable” Mechanisms

- We want to find the conditions for a mechanism so it would be “Desirable” and “Reasonable”.

- **Desirable**: The mechanism should minimize the efficiency loss when the users are *price anticipating* (low price of anarchy).
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The Search for “Desirable” and “Reasonable” Mechanisms

- We want to find the conditions for a mechanism so it would be “Desirable” and “Reasonable”.
- **Desirable:** The mechanism should minimize the efficiency loss when the users are *price anticipating* (low price of anarchy).
- **Reasonable:** The strategy space for each participant should be simple (of low dimension) - we'll focus on mechanisms where each user chooses a scalar from $\mathbb{R}^+$, meaning - low communication overhead.
- We’ll see that under certain assumptions, the PAM mechanism minimizes the efficiency loss when users are *price anticipating*.
Smooth Market Clearing Mechanism

Definition

A smooth market-clearing mechanism is a differential function $D : (0, \infty) \times [0, \infty) \mapsto \mathbb{R}^+$ such that

$\forall C > 0 \quad \forall R > 1 \quad \forall \theta \in (\mathbb{R}^+)^R \quad \exists p > 0$ (unique) s.t.:

$$\sum_{r=1}^{R} D(p, \theta_r) = C$$
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- Undefined if $\theta = 0$. 

A generalization of the demand function interpretation of the PAM. There we had $D(p, \theta) = \theta p$ and $p D(\theta) = \sum_{r} \theta_r / C$.
A **smooth market-clearing mechanism** is a differential function $D: (0, \infty) \times [0, \infty) \mapsto \mathbb{R}^+$ such that

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- Undefined if $\theta = 0$.
- A generalization of the demand function interpretation of the PAM. There we had $D(p, \theta) = \frac{\theta}{p}$ and $p_D(\theta) = \sum_r \theta_r / C$. 

---

**Definition**

**Smooth Market Clearing Mechanism**

**Outline**

- Introduction
- The Proportional Allocation Mechanism
- A Characterization Theorem
- The Vickery-Clarke-Groves (VCG) Approach
- Summary

**Motivation**

**Smooth Market Clearing Mechanism**

**General Class of Mechanisms** $\mathcal{D}$

$\mathcal{D}$ Characterization
Competitive Equilibrium for SMCM

Definition
Given a utility system \((C, R, U)\) and a SMCM \(D\), we say that a nonzero vector \(\theta\) is a competitive equilibrium if for \(\mu = p_D(\theta)\):

\[
\theta_r \in \arg \max_{\bar{\theta}_r \geq 0} \left[ U_r(D(\mu, \bar{\theta}_r)) - \mu D(\mu, \bar{\theta}_r) \right] \quad \forall r
\]

The utility you get minus the price you pay (same as payoff).
Competitive Equilibrium for SMCM

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Nash Equilibrium for SMCM

Definition

Given a utility system \((C, R, U)\) and a SMCM \(D\), we say that a nonzero vector \(\theta\) is a \textit{Nash equilibrium} if for all \(r\):

\[
\theta_r \in \arg \max_{\bar{\theta}_r \geq 0} Q_r(\bar{\theta}_r; \theta_{-r})
\]

where:

\[
Q_r(\theta_r; \theta_{-r}) = \begin{cases} 
U_r(D(p_D(\theta), \theta_r)) - p_d(\theta)D(p_D(\theta), \theta_r) & \theta \neq 0 \\
-\infty & \theta = 0
\end{cases}
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-\infty & \theta = 0
\end{cases}
\]

- Almost the same, but with \(-\infty\) when \(\theta = 0\).
Class of Mechanisms of Interest

**Definition**

The class $\mathcal{D}$ consists of all functions $D(p, \theta)$ such that:

- $D$ is a SMCM.
Class of Mechanisms of Interest

Definition

The class \( \mathcal{D} \) consists of all functions \( D(p, \theta) \) such that:

1. \( D \) is a SMCM.
2. For all \( C > 0 \) and for all \( U_r \in \mathcal{U} \) the user’s payoff is concave if he’s *price anticipating* (that is, \( Q_r(\theta_r; \theta_{-r}) \) is concave).
Class of Mechanisms of Interest

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The class $\mathcal{D}$ consists of all functions $D(p, \theta)$ such that:

- $D$ is a SMCM.
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- For all $p > 0$ and for all $d \geq 0$, there exists a $\theta > 0$ such that $D(p, \theta) = d$. 
Definition

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- $\forall p. \forall \theta. D(p, \theta) \geq 0$
Class of Mechanisms of Interest

Definition

The class $\mathcal{D}$ consists of all functions $D(p, \theta)$ such that:

- $D$ is a SMCM.
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- For all $p > 0$ and for all $d \geq 0$, there exists a $\theta > 0$ such that $D(p, \theta) = d$.
- $\forall p. \forall \theta. D(p, \theta) \geq 0$
Example of $D \in \mathcal{D}$

Example:

- Let $D(p, \theta) = \theta p^{-1/c}$ where $c \geq 1$. 
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- Note that for $c = 1$ we get the PAM mechanism.
- $\sum_r D(p_D(\theta), \theta_r) = C \Rightarrow p_D(\theta)^{-1/c} = C/\sum_r \theta_r \Rightarrow p_D(\theta) = (\sum_r \theta_r/C)^{1/c}$
Example of $D \in \mathcal{D}$

Example:

- Let $D(p, \theta) = \theta p^{-1/c}$ where $c \geq 1$.
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- Note that for $c = 1$ we get the PAM mechanism.
- $\sum_r D(p_D(\theta), \theta_r) = C \Rightarrow p_D(\theta)^{-1/c} = C / \sum_r \theta_r \Rightarrow p_D(\theta) = (\sum_r \theta_r / C)^{1/c}$
- Hence:

$$D(p_D(\theta), \theta_r) = \theta \left( \left( \frac{\sum_r \theta_r}{C} \right)^{1/c} \right)^{-1/c} = \frac{\theta}{\sum_r \theta_r} C$$
Example of $D \in \mathcal{D}$

**Conclusion:**
- For the given mechanism $D$, we saw that regardless of $c$, the market clearing allocations are chosen proportional to the bids.
Example of $D \in \mathcal{D}$

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Price of Anarchy for \( \mathcal{D} \)

- We’re interested in the worst case ratio between the aggregate utility at any Nash equilibrium to the optimal solution of \( \text{SYSTEM} \).
Price of Anarchy for $D$

- We’re interested in the worst case ratio between the aggregate utility at any Nash equilibrium to the optimal solution of $SYSTEM$.
- For every $D \in D$ we define a constant $\rho(D)$ as follows:

$$
\rho(D) = \inf \left\{ \frac{\sum_{r=1}^{R} U_r(D(p_D(\theta), \theta_r))}{\sum_{r=1}^{R} U_r(d_r)} \mid C > 0, R > 1, U \in U^R \right\}
$$

where $d$ solves $SYSTEM$, and $\theta$ is a Nash equilibrium.
We’re interested in the worst case ratio between the aggregate utility at any Nash equilibrium to the optimal solution of \( \text{SYSTEM} \).

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where \( d \) solves \( \text{SYSTEM} \), and \( \theta \) is a Nash equilibrium.

Models price of anarchy. \( \rho(D) \) gets it’s value when \( d \) is optimal for \( \text{SYSTEM} \) and \( \theta \) is the least efficient Nash equilibrium.
Characterization Theorem for $\mathcal{D}$

**Theorem**

Let $D \in \mathcal{D}$ be a smooth market-clearing mechanism. Then:

1. There exists a competitive equilibrium $\theta$. Furthermore, for any such $\theta$, the resulting allocation $d$ given by $d = D(p, \theta)$ solves SYSTEM.
2. There exists a concave, strictly increasing, differentiable, and invertible $B : (0, \infty) \mapsto (0, \infty)$ such that for all $p > 0$ and $\theta \geq 0$:
   
   $$D(p, \theta) = \theta B(p)$$

3. $\rho(D) \leq \frac{3}{4}$, and the bound is met with equality iff $D(p, \theta) = \Delta \theta / p$ for some $\Delta > 0$. 

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The Price of Anarchy and the Design of Scalable Resource Allocation Mechanisms
Characterization Theorem for \( \mathcal{D} \)

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Characterization Theorem for $D$

**Theorem**

Let $D \in \mathcal{D}$ be a smooth market-clearing mechanism. Then:

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The Price of Anarchy and the Design of Scalable Resource Allocation Algorithms
Characterization Theorem for $\mathcal{D}$

Proof.
Characterization Theorem for $\mathcal{D}$

**Proof.**

*Cause I say so.*
Characterization Theorem for $\mathcal{D}$

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Conclusions from the Characterization Theorem

- From the second part of the theorem:

\[
\sum_r D(p_D(\theta), \theta_r) = \frac{\sum_r \theta_r}{B(p_D(\theta))} = C \Rightarrow B(p_D(\theta)) = \frac{\sum_r \theta_r}{C}
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- Meaning: Every mechanism in \( \mathcal{D} \) chooses allocations in proportions to the bids.

- For each \( D \in \mathcal{D} \) we get the same allocations as PAM. The only difference is the market-clearing price.
Conclusions from the Characterization Theorem

- At least 25 percent efficiency loss.
Conclusions from the Characterization Theorem

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- The loss is minimized when a mechanism charges a user exactly their bid.
Motivation

- The mechanisms we offered had some restrictions:
  1. A simple strategy space (every user offers a price).
  2. A single market-clearing price.
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- But try keeping a low-dimensional strategy space.
The VCG Approach

- In the VCG class of mechanisms, each user’s strategy space is the predefined $U$. 
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- $d_r$ - the allocation of user $r$. $t_r$ - the payment of user $r$.
- The payoff is $U_r(d_r) - t_r$.
- The social objective can be written as $U_r(d_r) + \sum_{s \neq r} U_s(d_s)$. 
The VCG Approach

- Let \( \chi = \{ d \geq 0 : \sum_r d_r \leq c \} \) (feasible region for \( SYSTEM \)).
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$$d(\tilde{U}) \in \arg\max_{d \in \chi} \sum_r \tilde{U}_r(d_r)$$

The payments are structured so that:

$$t_r(\tilde{U}) = -\sum_{s \neq r} \tilde{U}_s(d_s(\tilde{U}) + h_r(\tilde{U} - r))$$

Different price per-unit for each user.

VCG mechanism $\rightarrow$ dominant strategy equilibrium where each user reveals his true utility function.

Because of the definition of $d(\tilde{U})$ we get full efficiency.
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- Different price per-unit for each user.
- VCG mechanism \( \rightarrow \) dominant strategy equilibrium where each user reveals his true utility function.
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Presenting the Scalar Strategy VCG(SSVCG)

- Too much communication overhead (users need to pass a full description of the utility).
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- Instead, users choose from a given single parameter of utility functions $\overline{U}(.; \theta)$. 

Assumption 2:
- For every $\theta > 0$, the function $U(d; \theta)$ belongs to $\mathcal{U}$ and also strictly concave.
- For every $\gamma \in (0, \infty)$ and $d \geq 0$, there exists a $\theta > 0$ such that $U'(d; \theta) = \gamma$. 

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The Price of Anarchy and the Design of Scalable Resource Allocation Mechanisms
Presenting SSVCG

Given $\theta$, the mechanism chooses allocation:

$$d(\theta) = \arg \max_{d \in \chi} \sum_r U(d_r; \theta_r)$$
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- The payment is:

$$t_r(\theta) = - \sum_{s \neq r} U(d_s(\theta); \theta_s) + h_r(\theta_{-r})$$

where $h_r$ depends only on strategies $\theta_{-r}$. 
Payoff and Nash Equilibrium

- The payoff is defined:

\[ P_r(d_r(\theta), t_r(\theta)) = U_r(d_r(\theta)) + \sum_{s \neq r} \bar{U}(d_s(\theta); \theta_s) - h_r(\theta_{-r}) \]
The payoff is defined:

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\( \theta \) is a Nash equilibrium if for all \( r \):

\[ P_r(d_r(\theta), t_r(\theta)) \geq P_r(d_r(\theta', \theta_{-r}), t_r(\theta', \theta_{-r})) \quad \forall \theta' > 0 \]
Lemma

The vector $\theta$ is a Nash equilibrium iff for all $r$:

$$d(\theta) \in \arg \max_{d \in \chi} \left[ U_r(d_r) + \sum_{s \neq r} \overline{U}(d_s; \theta_s) \right]$$
Connecting Nash Equilibrium and Utilities

**Lemma**

The vector $\theta$ is a Nash equilibrium iff for all $r$:

$$d(\theta) \in \arg \max_{d \in X} \left[ U_r(d_r) + \sum_{s \neq r} \bar{U}(d_s; \theta_s) \right]$$

**Proof.**

$\Leftarrow$:
Connecting Nash Equilibrium and Utilities

**Lemma**

The vector $\theta$ is a Nash equilibrium iff for all $r$:

$$d(\theta) \in \arg \max_{d \in \chi} \left[ U_r(d_r) + \sum_{s \neq r} \overline{U}(d_s; \theta_s) \right]$$

**Proof.**

$\Leftarrow$:

- Fix user $r$. Since $\theta_r$ doesn’t effect $h_r$ in the payoff, user $r$ will choose $\theta_r$ to maximize $U_r(d_r(\theta)) + \sum_{s \neq r} \overline{U}(d_s(\theta); \theta_s)$.
Lemma

The vector $\theta$ is a Nash equilibrium iff for all $r$:

$$d(\theta) \in \arg \max_{d \in \chi} \left[ U_r(d_r) + \sum_{s \neq r} \bar{U}(d_s; \theta_s) \right]$$

Proof.

$\Leftarrow$:

- Fix user $r$. Since $\theta_r$ doesn’t effect $h_r$ in the payoff, user $r$ will choose $\theta_r$ to maximize $U_r(d_r(\theta)) + \sum_{s \neq r} \bar{U}(d_s(\theta); \theta_s)$.
- We assume that the following holds for all $r$ in $\theta$. 

Connecting Nash Equilibrium and Utilities

proof-cont.

⇒:

\[\text{Suppose } \theta \text{ is a NE.} \]

\[\text{Suppose there exists } r \text{ for which } \frac{d(\theta)}{\in \arg \max_{d \in \chi} \left[ U_r(d) + \sum_{s \neq r} U(d_s; \theta_s) \right].} \]

\[\chi \text{ is compact } \rightarrow \exists \text{ optimal } d^* \text{ for the latter.} \]

\[\text{From that, } U'_r(d^*_r) + \sum_{s \neq r} U(d^*_s; \theta_s) = 0. \]

\[\text{If user } r \text{ chooses } \theta'_r > 0 \text{ such that } U'_r(d^*_r) = U'_r(d^*_r) \] then \[d^* \text{ is also the (unique) optimal allocation for } (\theta'_r, \theta - r). \]
proof-cont.

⇒:

- Suppose $\theta$ is a NE.
Connecting Nash Equilibrium and Utilities

proof-cont.

\( \Rightarrow: \)

- Suppose \( \theta \) is a NE.
- Suppose there exists \( r \) for which
  \[ d(\theta) \notin \arg \max_{d \in \chi} \left[ U_r(d_r) + \sum_{s \neq r} U(d_s; \theta_s) \right]. \]
proof-cont.

\[ \Rightarrow: \]

1. Suppose \( \theta \) is a NE.
2. Suppose there exists \( r \) for which
   \[
   d(\theta) \notin \arg \max_{d \in \chi} \left[ U_r(d_r) + \sum_{s \neq r} U(d_s; \theta_s) \right].
   \]
3. \( \chi \) is compact \( \rightarrow \) exists optimal \( d^* \) for the latter.
proof-cont.

\[ \Rightarrow: \]

- Suppose \( \theta \) is a NE.
- Suppose there exists \( r \) for which
  \[ d(\theta) \notin \arg \max_{d \in \chi} \left[ U_r(d_r) + \sum_{s \neq r} \overline{U}(d_s; \theta_s) \right]. \]
- \( \chi \) is compact \( \rightarrow \) exists optimal \( d^* \) for the latter.
- From that, \( U_r'(d^*_r) + \sum_{s \neq r} \overline{U}(d^*_s; \theta_s) = 0. \)
proof-cont.

⇒:

- Suppose $\theta$ is a NE.
- Suppose there exists $r$ for which $d(\theta) \notin \arg \max_{d \in \chi} \left[ U_r(d_r) + \sum_{s \neq r} U(d_s; \theta_s) \right]$.  
- $\chi$ is compact $\rightarrow$ exists optimal $d^*$ for the latter.
- From that, $U'_r(d^*_r) + \sum_{s \neq r} U(d^*_s; \theta_s) = 0$.
- If user $r$ chooses $\theta'_r > 0$ such that $\overline{U}'(d^*_r) = U'_r(d^*_r)$ then $d^*$ is also the (unique) optimal allocation for $(\theta'_r, \theta_{-r})$. 

Connecting Nash Equilibrium and Utilities

⇒-cont.

- We get:
Connecting Nash Equilibrium and Utilities

⇒-cont.

We get:

\[ P_r(d_r(\theta), t_r(\theta)) = U_r(d_r) + \sum_{s \neq r} U(d_s; \theta_s) + h_r(\theta_{-r}) \]
Connecting Nash Equilibrium and Utilities

⇒-cont.

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⇒-cont.

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\]

\[
= U_r(d_r(\theta'_r, \theta_{-r})) + \sum_{s \neq t} U(d_s(\theta'_r, \theta_{-r}); \theta) + h_r(\theta_{-r})
\]
Connecting Nash Equilibrium and Utilities

⇒-cont.

- We get:

\[ P_r(d_r(\theta), t_r(\theta)) = U_r(d_r) + \sum_{s \neq r} U(d_s; \theta_s) + h_r(\theta_{-r}) \]
\[ < U_r(d^*_r) + \sum_{s \neq r} U(d^*_s; \theta_s) + h_r(\theta_{-r}) \]
\[ = U_r(d_r(\theta'_r, \theta_{-r})) + \sum_{s \neq t} U(d_s(\theta'_r, \theta_{-r}); \theta_s) + h_r(\theta_{-r}) \]
\[ = P_r(d_r(\theta'_r, \theta_{-r}), t_r(\theta'_r, \theta_{-r})) \]
Connecting Nash Equilibrium and Utilities

\( \Rightarrow \)-cont.

- We get:

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\]

\[
= P_r(d_r(\theta'_r, \theta_{-r}), t_r(\theta'_r, \theta_{-r}))
\]

- In contradiction of \( \theta \) being a NE.
Corollary

For any SSVCG mechanism, there exists an efficient Nash equilibrium $\theta$ defined as follows: Let $d^S$ be an optimal solution to SYSTEM. Each user $r$ chooses $\theta_r$ so that $U'(d^S_r; \theta_r) = U'_r(d^S_r)$. The resulting allocation satisfies $D(\theta) = d^S$. 

Proof. Each user $r$ can choose $\theta_r$ so that $U'(d^S_r; \theta_r) = U'_r(d^S_r)\sum_r U'(d^S_r; \theta_r) = \sum_r U'_r(d^S_r)$. By taking derivatives, $d^S \in \arg\max d \in \chi [U_r(d^S_r) + \sum_{s \neq r} U(d^S_s; \theta_s)]$. Hence, $\theta$ is a NE.
Efficient Price of Stability

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For any SSVCG mechanism, there exists an efficient Nash equilibrium $\theta$ defined as follows: Let $d^S$ be an optimal solution to SYSTEM. Each user $r$ chooses $\theta_r$ so that $\overline{U}'(d^S_r; \theta_r) = U'_r(d^S_r)$. The resulting allocation satisfies $D(\theta) = d^S$.

Proof.

- Each user $r$ can $\theta_r$ so that $\overline{U}'(d^S_r; \theta_r) = U'_r(d^S_r)$
Efficient Price of Stability

**Corollary**

For any SSVCG mechanism, there exists an efficient Nash equilibrium \( \theta \) defined as follows: Let \( d^S \) be an optimal solution to SYSTEM. Each user \( r \) chooses \( \theta_r \) so that \( U'(d^S_r; \theta_r) = U'_r(d^S_r) \). The resulting allocation satisfies \( D(\theta) = d^S \).

**Proof.**

- Each user \( r \) can \( \theta_r \) so that \( U'(d^S_r; \theta_r) = U'_r(d^S_r) \)
- \( \sum_r U'(d^S_r; \theta_r) = \sum_r U'_r(d^S_r) = 0 \)
Corollary

For any SSVCG mechanism, there exists an efficient Nash equilibrium $\theta$ defined as follows: Let $d^S$ be an optimal solution to SYSTEM. Each user $r$ chooses $\theta_r$ so that $\overline{U}'(d^S_r; \theta_r) = U'_r(d^S_r)$. The resulting allocation satisfies $D(\theta) = d^S$.

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- By taking derivatives, $d^S \in \arg\max_{d \in \chi} \left[U_r(d_r) + \sum_{s \neq r} \bar{U}(d_s; \theta_s)\right]$
- Hence, $\theta$ is a NE.
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- It’s also possible to construct such an example if a user declares a high marginal utility compared to the other users.
- On the other hand, if we set $U'_r = \infty$ for all users, the NE achieves full efficiency.
- Intuitively, for an SSVCG mechanism being efficient, we would like the several users competing with each other.
Summery of Subjects Investigated

**Single Market-Clearing Price:**

- A naive approach of proportionally allocating the resource is pretty efficient.
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**VCG Approach:**
- Taking the VCG approach allows us reach full efficiency.
- Using a quasi-VCG mechanism where the communication overhead is small gives us no efficiency loss when we take the optimal NE.
- The SSVCG can be tweaked so that the price of anarchy would be minimized.