On-line and Approximation Algorithms **Spring Semester, 2021/22**

Exercise 2: April 4, 2022

Lecturer: Prof. Yossi Azar

Write short but full and accurate answers. Each solution should appear on a **separate** page and each of its parts should not exceed a page.

1. Consider the on-line load balancing problem on m machines (m is even). Job i has a subset of machines S_i of size $m/2$ and two values w_i and p_i. Assigning job i is assigned to a machine $j \in S_i$ increases the load of j by w_i where assigning job i to a machine $j \notin S_i$ increases the load of j by p_i. The goal is to minimize the maximum load.
   (a) Design a $3 - \frac{2}{m}$ competitive algorithm.
   (b) Show a lower bound of 2 for any (even) m.
2. Consider the on-line load balancing problem of tasks on m machines in the restricted assignment model. Consider the case where for each i the set of machines which is associated with job i is $[1 \ldots k_i]$ for some $1 \leq k_i \leq m$. The goal is to minimize the maximum load.
   (a) Design a constant competitive algorithm.
   (b) Show a lower bound of 2 for deterministic algorithms, for any $m \geq 6$, using only unit jobs. Hint: start with $m = 6$. (you will get a partial credit for a lower bound of 11/6.)
   (c) Show a lower bound of $2 - O(\frac{1}{m})$ for randomized algorithms, for any m, using only unit jobs.
3. Consider the on-line load balancing problem in the restricted assignment model where both the algorithm and the optimum are allowed to split jobs (in any way).
   (a) Define the natural "water level" algorithm and show that it is at most $\log m + 2$ competitive.
   (b) Show a lower bound of $H_m = 1 + 1/2 + \ldots + 1/m$ on the competitive ratio of any algorithm for the problem even if the optimum does not split jobs and all jobs have unit size.
4. We are given a connected graph $G = (V, E)$. All edges have unit capacity. At round i we receive a request to connect s_i to t_i with bandwidth q_i on a path Q_i or with bandwidth r_i on a path R_i (Q_i and R_i are given disjoint paths from s_i to t_i).
   (a) Assume we must assign the request to one of the two paths and the goal is to minimize the maximum load over the edges. Design an $O(\log(|E|)$ competitive algorithm.
   (b) We may either assign the request to one of the two paths or reject it and pay $\beta_i > 0$ for the rejection. The goal is to minimize the sum of the costs of all rejected requests plus the maximum load. Design an $O(\log(|E|)$ competitive algorithm.

   Remark: Do not re-prove theorems proved in class for (a)(b).
5. Consider the problem of on-line load balancing of jobs on kr machines that are partitioned into k groups, each consists of r identical machines. Job i has a weight vector of length k were w_{ij} (for $1 \leq j \leq k$) is the load that the job adds to a machine if it is assigned to a machine in the group j.
   (a) Show an $O(\log k)$ competitive algorithm where the goal is to minimize the maximum group volume (group volume is the total load of machines in a group).
   (b) Show an $O(\log k)$ competitive algorithm for minimizing the maximum load over all machines (Note that a solution that only balances the volume between groups cannot be transformed in general into a solution that balances the loads).
   (c) Show a lower bound for minimizing the maximum load over all machines of $\Omega(\log k)$ for any k and r.

Exercise # 2 is due April 25, 2022 at 2PM .