Write short but full and accurate answers. Each solution should appear on a separate page and each of its parts should not exceed a page.

1. Consider the on-line load balancing problem on $m$ machines ($m$ is even). Job $i$ has a subset of machines $S_i$ of size $m/2$ and two values $w_i$ and $p_i$. Assigning job $i$ is assigned to a machine $j \in S_i$ increases the load of $j$ by $w_i$ where assigning job $i$ to a machine $j \notin S_i$ increases the load of $j$ by $p_i$. The goal is to minimize the maximum load.

   (a) Design a $3 - \frac{2}{m}$ competitive algorithm.
   (b) Show a lower bound of 2 for any (even) $m$.

2. Consider the on-line load balancing problem of tasks on $m$ machines in the restricted assignment model. Consider the case where for each $i$ the set of machines which is associated with job $i$ is $[1 \ldots k_i]$ for some $1 \leq k_i \leq m$. The goal is to minimize the maximum load.

   (a) Design a constant competitive algorithm.
   (b) Show a lower bound of 2 for deterministic algorithms, for any $m \geq 6$, using only unit jobs. Hint: start with $m = 6$. (you will a partial credit for a lower bound of $11/6$.)
   (c) Show a lower bound of $2 - O(1/m)$ for randomized algorithms, for any $m$, using only unit jobs.

3. Consider the on-line load balancing problem in the restricted assignment model where both the algorithm and the optimum are allowed to split jobs (in any way).

   (a) Define the natural "water level" algorithm and show that it is at most $\log m + 2$ competitive.
   (b) Show a lower bound of $H_m = 1 + 1/2 + \ldots + 1/m$ on the competitive ratio of any algorithm for the problem even if the optimum does not split jobs and all jobs have unit size.

4. Consider an on-line routing problem on a connected graph $G = (V, E)$ with unit capacities. The request $i$ consists of vertices $s_i, t_{i,1}, \ldots t_{i,k}$ and real numbers $w_{i,1}, \ldots w_{i,k}$. To serve the $i$’th request you need to choose some $r$, $1 \leq r \leq k$ and allocate one path from $s_i$ to $t_{i,r}$ of bandwidth $w_{i,r}$. Design an $O(\log |E|)$ competitive algorithm for minimizing the maximum load.

5. Consider the problem of on-line load balancing of jobs on $kr$ machines that are partitioned into $k$ groups, each consists of $r$ identical machines. Job $i$ has a weight vector of length $k$ were $w_{ij}$ (for $1 \leq j \leq k$) is the load that the job adds to a machine if it is assigned to a machine in the group $j$.

   (a) Show an $O(\log k)$ competitive algorithm where the goal is to minimize the maximum group volume (group volume is the total load of machines in a group).
   (b) Show an $O(\log k)$ competitive algorithm for minimizing the maximum load over all machines (Note that a solution that only balances the volume between groups cannot be transformed in general into a solution that balances the loads).
   (c) Show a lower bound for minimizing the maximum load over all machines of $\Omega(\log k)$ for any $k$ and $r$.

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Exercise # 2 is due April 23, 2014 at 4PM.