On-line and Approximation Algorithms	**Spring Semester, 2021/22**

Exercise 1: Mar 14, 2022

Lecturer: Prof. Yossi Azar

Write short but full and accurate answers. Each solution should appear on a **separate** page and each of its parts should not exceed a page.

1. Suppose we have a set of requests over the real continuous time. Each request arrives at an arbitrary time (which is a real number). An algorithm may provide a service point at any time (a real number - not necessarily immediately after the request). The response time of a request is the time from its arrival until the next service point. The goal is to minimize the number of service points plus the total response time of all requests. Note that a service point serves all waiting requests simultaneously (i.e. it serves all requests that arrive after the previous service point).
   (a) Design a 2 competitive deterministic algorithm.
   (b) Show a lower bound of 2 (more precisely $2-\epsilon$ for any positive ϵ) even if an additive constant is allowed (i.e. $A(\sigma)$ should go to infinity). Hint: create a request immediately after any service point.
   (c) How would your answers to (a) and (b) change if each service point can serve at most k requests for some fixed $k > 1$.

2. Consider a paging problem of pages of different sizes. Page i has size x_i which is an integer number between 1 and k. The memory can hold pages of total size k. The cost of bringing a page i to the memory is equal to its size. Once you remove part of a page P from the memory then P becomes unavailable in the memory and you would need to bring the whole page P at the next request for P.
   (a) Design a deterministic k competitive algorithm.
   (b) Assume that for all i we have $x_i = r$. Design a $\lfloor k/r \rfloor$ competitive deterministic algorithm.

3. Suppose you need to find an unknown point (x_0, y_0) in the Euclidean plane. You start at the origin and you can move in any continues curve in the plane. You find your point when you reach a point (x_0, y) for any y or a point (x, y_0) for any x (you do not need to reach (x_0, y_0)). Your goal is to minimize the curve's length. Algorithms are deterministic.
   (a) Show a $\sqrt{2}$ competitive algorithm if it is known that $x_0 \geq 0$ and $y_0 \geq 0$.
   (b) Show a lower bound of $\sqrt{2}$ for the case where $x_0 \geq 0$ and $y_0 \geq 0$.
   (c) Show a $9\sqrt{2}$ competitive algorithm for arbitrary x_0 and y_0 (additive constant is allowed).
   (d) Design an algorithm for the case of $x_0 \geq 0$ and arbitrary y_0 whose competitive ratio is strictly below 10 (additive constant is allowed).

4. Consider the following variant of the list update problem of a list of n elements. At any request the algorithm (and especially the adversary) is allowed to take any (up to) 3 elements in the segment between the head of the list up to the currently requested element and move them for free anywhere in that segment. This is in addition to the optional moving of the requested element towards the head of the list for free.
   (a) Show that MoveToFront is 5 competitive.
   (b) Show a lower bound for any deterministic algorithm of $4-O(1/n)$ even if an additive constant is allowed (i.e. $A(\sigma)$ should go to infinity). Note: the optimum is not allowed to make paid exchanges that involve elements located after the currently requested element.
   Remark: A lower bound of $3 - O(1/n)$ will get a partial credit.

5. Consider the ski rental problem (1 to rent, M to buy) with the following frequent skier program. After $(1+\gamma)M$ times (for some fixed real number $\gamma > 0$, for example, $\gamma = 0.7$ or $\gamma = 2.3$) you have rented the skis you get them for free.
   (a) Find the best deterministic strategy and its competitive ratio (as a function of γ).
   (b) Show a tight lower bound for any deterministic algorithm for the problem.
   (c) Find the best randomized strategy and its competitive ratio. Also compute it for $\gamma = 0.8$.

Exercise # 1 is due Mar 28, 2022 at 2PM .