Algorithmic Methods **Spring Semester, 2022/23**

Exercise 1: April 17, 2023

Lecturer: Prof. Yossi Azar

Write short but full and accurate answers. Each question should start on a new **separate** page and each of its parts should not exceed a page.

1. Let x be a feasible point for LP in the standard form Min $c^t x$ for $Ax = b$ and $x \geq 0$. Let $Z = \{i | x_i = 0\}$. Prove that x is an optimal solution if and only if the optimal value of the following LP is 0: Min $c^t y$ for $Ay = 0$ and $y_i \geq 0$ for all $i \in Z$.

2. Let $a_1, a_2, \ldots a_m \in R^n$ such that $m > n+1$. Assume that for the set of m inequalities $a_i^t x \leq b_i$ there is no feasible solution. Prove that there are $n+1$ inequalities out of the m which are not feasible.

3. (a) Prove that if an LPS (standard form LP) has a non-degenerate vertex which is an optimal solution then the dual problem has a unique optimal solution.
   (b) Does the above remain true if the LPS has an optimal solution (not necessarily a vertex) with m variables of non-zero values (the LPS has m equations) ?

4. We are given a set of n points in R^2, $(x_1, y_1), \ldots, (x_n, y_n)$. Our goal is to find a function of the type $f(x) = ax^2 + b2^x + c$ such that $\sum_{i=1}^{n} |f(x_i) - y_i|$ is minimum.
   (a) Show how find such a function f given a polynomial time algorithm for LP.
   (b) Show how to find such a function f if the goal is to minimize $\max_{1 \leq i \leq n} (f(x_i) - y_i)^2$.

5. We are given n jobs that needs to be assigned to m machines. The load job i creates if assigned to machine j is $w_{i,j}$. The load of a machine is the sum of the loads of the jobs assigned to it. A job can be split and each part can be assigned to a different machine (the load created by a fraction of a job is equal to that fraction times the load created by the whole job). Assume we are given a polynomial time algorithm for LP.
   (a) Find an assignment that maximizes the minimum load over all machines.
   (b) Find an assignment which minimizes the sum of the loads of the two most loaded machines.
   (c) Show that there is an assignment that minimizes the sum of the loads of all machines such that no job needs to be split.

Exercise # 1 is due April 30, 2023 at 11pm