Write short but full and accurate answers. Each question should start on a new separate page and each of its parts should not exceed a page.

1. We are given \( n \) jobs and \( m \) unrelated machines. The load of job \( i \) on machine \( j \) is \( w_{ij} \). The load of a machine is the sum of the weights of the jobs assigned to it. In contrast to the standard problem here each job \( i \) has two copies and they should be assigned exactly to TWO different machines say \( j_1 \neq j_2 \) (then the load of \( j_1 \) would increase by \( w_{ij_1} \) and the load \( j_2 \) would increase by \( w_{ij_2} \)). The goal is to minimize the maximum load.

(a) Write the appropriate LP formulation.
(b) Round the LP and provide a 2 approximation algorithm. (recall that the two machines each job is assigned to must be different)

2. Suppose we are given a regular graph \( G = (V, E) \) of degree \( \Delta \). Each vertex has a different i.d (which initially is unknown to the others) between 0 to \( 2^n - 1 \) where \( |V| = n \). Recall that a local algorithm with \( k \) rounds is an algorithm where each vertex decides on its output after \( k \) synchronized communication rounds with its neighbors. Find a local algorithm that colors the graph in \( \Delta + 1 \) colors in \( \log^* n + 2^{O(\Delta)} \) rounds.

Remark: a solution in \( \log^* n + 2^{O(\Delta \log \Delta)} \) rounds will receive almost all points.

3. Assume we are given a rooted tree where vertices may have arbitrary degrees. Each vertex has a unique label between 1 and \( n \). Each vertex has to choose a color based only on local information such that the resulting coloring is a valid one. Each vertex has the knowledge of which out of its incident edges is directed toward the root (the root knows it is the root).

(a) Claim an upper bound on the number of rounds required to color the rooted tree in 6 colors.
(b) Given a 6 vertex-coloring of the rooted tree transform it to a 3-coloring in constant number of rounds. Hint: transform the 6-coloring into a new 6-coloring such that each vertex has neighbors of only 2 colors.

4. We are given a tree and requests \((s_i, t_i)\) with a bandwidth \( w_i \) and a value \( v_i \) for \( 1 \leq i \leq n \). The goal is to maximize the total value of requests in a feasible subset.

(a) Design a 3 approximation algorithm where a feasible set is a subset of the requests with total bandwidth at most 1 on each vertex.
(b) Modify the algorithm and the proof to design a 6 approximation algorithm where a feasible set is a subset of the requests with bandwidth at most 1 on each edge.

5. We are given a graph \( G = (V, E) \) with cost on the vertices \( c : V \to R^+ \) and penalty on the edges \( p : E \to R^+ \). A partial vertex cover is any subset \( S \subseteq V \) and its cost is \( \sum_{v \in S} c(v) + \sum_{e \in F} p(e) \) where \( F \) is the set of edges \((u, v)\) for which neither \( u \in S \) nor \( v \in S \).

(a) Form an integer LP, relaxed it and round it deterministically to get 3-approximation polynomial time algorithm.
(b) Use local ratio to get 2 approximation polynomial time algorithm (do not forget to characterize the instances for each the optimal value equals 0).

Exercise # 3 is due Jun 7, 2021.