Write short but full and accurate answers. Each question should start on a new page and each of its parts should not exceed a page. Two pages A4 (both sides each) are allowed.

1. We are given \( r + 1 \) groups of machines, group \( j \) for \( 0 \leq j \leq r \) includes \( m_j \) machines. Jobs arrive one by one, job \( i \) has weight \( w_i \) and type \( k_i \) where \( 1 \leq k_i \leq r \). Job \( i \) of type \( k_i \) can be assigned to machines only in group \( k_i \) and in group 0. The goal function is to minimize the maximum load over all machines.

   (a) Design a constant competitive deterministic algorithm. Hint: assume first that the value of \( \text{OPT} \) is known and then prefer group \( k > 0 \) over group 0.

   (b) Assume that all jobs are of unit size jobs (\( w_i = 1 \) for all \( i \)) and all groups contain one machine (\( m_j = 1 \) for all \( j \)). Show a lower bound of \( 3/2 \) for any deterministic algorithm.

2. Consider scheduling of unit size packets over time. A packet \( i \) arrives at integer time \( a_i \) and has a real value \( v_{i,t} \geq 0 \) for each integer \( t \geq a_i \). A packet \( i \) may be scheduled only at an integer time slot \( t \geq a_i \) which yields a real non-negative value \( v_{i,t} \). Also for each \( i \) the values are non-increasing sequence, i.e., \( v_{i,t+1} \leq v_{i,t} \) for all \( t \geq a_i \). Each packet can be transmitted at most once and only one packet may be transmitted at each time unit. The algorithm needs to choose which packet to transmit at each time slot. The goal is to maximize the total value of the transmitted packets.

   (a) Design a 2 competitive algorithm.

   (b) Show a lower bound of \( \sqrt{2} \) for any deterministic algorithm. Hint: use positive values only for two or three packets and only two time slots. Remark: smaller lower bound gets partial points.

   (c) Now assume that each packet needs 3 consecutive slots and once a transmission of a packet starts it cannot be interrupted. Show how to get a constant competitive randomized algorithm.

3. We are given a connected graph \( G = (V,E) \). All edges have unit capacity. At round \( i \) we receive a request to connect \( s_i \) to \( t_i \) on either a specific path \( Q_i \) or another specific path \( R_i \) with bandwidth of either \( a_i \) or \( b_i \) such that \( 0 < a_i < b_i < \frac{1}{\log(2|V|+2)} \). The algorithm needs to decide on one of the two paths and on one of the two bandwidths (four possibilities) or reject the request (fifth possibility). The goal is to maximize total throughput (total accepted bandwidth), while maintaining the capacity constraints. Design \( O(\log(|V|)) \) competitive algorithm.

   **Remark:** state the algorithm precisely. Recall that there are two lemmas one about the request accepted by the online and one about the request rejected by the online and accepted by \( \text{OPT} \) and then the two lemma are combined. You do not need to reprove the lemma about the accepted requests, however, you do need to prove everything else.

4. At time \( t = 0 \) two breakpoints are located at time 0. Time advances continuously. At each point in time \( t \) the algorithm needs to decide whether to delete the earlier breakpoint (among the two) and move it to the current time \( t \). Hence, at any given time (snapshot) the 2 breakpoints divide the time into 3 segments (each of length 0 or more). Define \( ON(t) \) as the length of the longest segment at time \( t \). Observe that \( \text{OPT}(t) = t/3 \). The competitive ratio is the maximum over \( t \) of \( ON(t)/\text{OPT}(t) \) (we ignore small \( t \) as is explained below). Observe that any algorithm can be described as an infinite sequence of breakpoint-moving times \( 0 < x_0 < x_1 < x_2 \ldots \).

   (a) Define the competitive ratio as a function of the sequence \( \{x_i\}_{i \geq 0} \). Remark: Assume \( x_0 = 1 \) and \( x_1 \) is finite and prove ONLY for \( t \geq x_1 \) (ignore the range \( t < x_1 \)). Hint: Show that the worst ratio occurs at time \( x_i \) for some \( i \).

   (b) Find a sequence \( \{x_i\}_{i \geq 0} \) that achieves the best competitive ratio (upper bound and lower bound). Hint: the algorithm may use geometric sequence. The lower bound should work for any algorithm.

The duration of the exam is 3 hours. GOOD LUCK