Write short but full and accurate answers. Each question should start on a new page and each of its parts should not exceed a page. Solving 4 questions results in score 90. Two pages A4 (both sides each) are allowed.

1. Every morning you are told if you need to drive to work. If yes then you need a parking permit that costs 1 per day. Every night you can buy a long term permit that lasts for 20k consecutive days starting from the next morning and costs k (where k is one given fixed integer). The long term permit expires in 20k days even if it is only partially (or not at all) used.

(a) Find the best competitive deterministic algorithm you can.
(b) Show that no deterministic algorithm can achieve a better competitive ratio.
(c) Assume that the long term permit can be bought only on nights of index t which is zero mod k.

Design c competitive algorithm where $c = (3 + \sqrt{5})/2$ (larger constant - partial points).

2. Consider the following variant of the list update problem. At any request the algorithm (as well as OPT) is allowed to take one element in the segment between the head of the list up to the currently requested element and move it for free anywhere in that segment. This is in addition to the option of moving the requested element towards the head of the list for free. Algorithm 1.5MoveToFront is defined in this model as follows. Given a request to an element $x$ at position $i$ it moves $x$ to the front of the list. In addition, if $i > 1$ it moves the element $y$ at position $i - 1$ to position $\lceil i/2 \rceil + 1$.

(a) Show that 1.5MoveToFront is 6 competitive.
(b) Show that the algorithm is 4 competitive if $x$ is moved to the front and $y$ is moved to position $[3i/4] + 1$ (instead of $[i/2] + 1$).

3. Consider the problem of on-line load balancing of jobs on $n$ machines that are partitioned into $k$ groups, where group $j$ consists of $n_j$ identical machines ($1 \leq j \leq k$). Job $i$ has a weight vector of length $k$ were $w_{ij}$ (for $1 \leq j \leq k$) is the load that the job adds to a machine if it is assigned to any machine in the group $j$. It is also known that $n_j/n_j' \leq \log k$ for all $j$ and $j'$ (the base of the log is 2).

(a) Show that $n_j \geq n/(k \log k)$ for all $j$.
(b) Show an $O(\log^2 k)$ competitive algorithm for minimizing the maximum load over all machines.
(c) For $k=2$ show a 3 competitive algorithm.

4. Consider admission control for the edge disjoint paths problem on a line (interval scheduling). The goal is to maximize the sum of the lengths of the accepted intervals. Given a new interval $I$ denote by $I_R$ ($I_L$, respectively) the interval that intersects its right point (left point, respectively) if exists. The algorithm SUM accepts the new interval $I$ if and only if $|I_R| + |I_L| \leq \alpha|I|$ (if an interval does not exist then its length is 0). If $I$ is accepted then it preempts $I_R$ and $I_L$ as well as all intervals contained in $I$.

(a) Prove that the algorithm SUM is constant competitive for $\alpha = 1/2$ (please define exactly the appropriate tree).
(b) Show an example that for $\alpha = 1$ the algorithm SUM is not constant competitive.

5. Suppose we have one machine and jobs released over time: job $i$ is released at time $r_i$, has size $w_i$, benefit $b_i$ and deadline $d_i$. Jobs are allowed to be preempted (i.e., interrupted and later resumed) and/or partially executed (as long as it is before the deadline). Denote by $p_i (\leq w_i)$ the total time job $i$ was processed before its deadline. Our goal is to maximize $\sum_i \frac{p_i}{w_i} b_i$. Assume that $r_i, w_i, d_i$ (but not $b_i$) are integers and the algorithm as well as OPT are allowed to preempt jobs only at integer times.

(a) Design a 2 competitive algorithm for the problem.
(b) Show a lower bound of $\sqrt{2}$ (smaller constant - partial points).

The duration of the exam is 3 hours. GOOD LUCK