Write short but full and accurate answers. Each question should start on a new page and each of its parts should not exceed a page. Solving 4 questions results in score 90. Two pages A4 (both sides each) are allowed.

1. Consider the following variant of ski rental problem. As usual it costs 1 to rent per time unit. At any time you can pay $M$ once and from that time on you pay $0 \leq \beta \leq 1$ per time unit.

(a) Find the best competitive deterministic algorithm you can.
(b) Show that no deterministic algorithm can achieve a better competitive ratio.
(c) Show that renting indefinitely is not an optimal competitive algorithm unless $\beta = 1$.

2. Consider the k-server on two disjoint segments $[a, b]$ and $[c, d]$, each of length $M/2$ for $M > 0$. Some of the $k$ servers are in one segment and the others are on the second segment. A server can jump from the endpoint $a$ to the endpoint $c$ or vice versa at a cost of $M$.

(a) Design a $k$-competitive algorithm for the problem.
(b) Design a $2k$-competitive algorithm assuming that a server can jump from any point in one segment to any point in the other segment at a cost of $M$ (Hint: think on the power of OPT).

3. Consider the on-line load balancing problem in the restricted assignment model where all jobs have unit size. The goal is to minimize the maximum load.

(a) Design an algorithm which is at most $\log m + 2$ competitive.
(b) Assume that all jobs are of sizes between 1 and $1 + \epsilon$ for some fixed $\epsilon > 0$. Design $(1 + \epsilon) \log m + O(1)$ competitive algorithm.

4. We are given a connected graph $G = (V, E)$. All edges have unit capacity. At round $i$ we receive a request to connect $s_i$ to $t_i$ with bandwidth $q_i$ on a path $Q_i$ or with bandwidth $r_i$ on a path $R_i$ ($Q_i$ and $R_i$ are given disjoint paths from $s_i$ to $t_i$).

(a) Assume we must assign the request to one of the two paths and the goal is to minimize the maximum load over the edges. Design an $O(\log(|V|))$ competitive algorithm.
(b) We may either assign the request to one of the two paths or reject it and pay $\beta_i > 0$ for the rejection. The goal is to minimize the sum of the costs of all rejected requests plus the maximum load. Design an $O(\log(|V|))$ competitive algorithm (Hint: slightly change the goal function).
(c) Assume $q_i = r_i$ and we may either assign the request to both paths simultaneously (one path is not enough) or reject it. Also $r_i = q_i \leq 1/\log(2|V|)$ . Our goal is to maximize the total throughput while maintaining the capacity constraints. Design an $O(\log(|V|))$ competitive algorithm.

Remark: Do not re-prove theorems proved in class for (a)(b)(c).

5. Consider scheduling of packets problem over time. Request $i$ arrives at time $a_i$ has deadline $d_i$, size $w_i$ slots (all numbers are integers) and a value $v_i$. To get the value $v_i$ for the packet $i$ one need to transmit the packet on $w_i$ consecutive slots before the deadline. A packet can be preempted (at integer times) but once preempted it is lost for ever (we cannot continue or restart the transmission).

(a) Design a constant competitive preemptive algorithm if $w_i = k$ for all $i$
(b) Modify the algorithm and the proof if all packets are of sizes of at least $k$ and at most $2k$.
(c) Design $O(\log k)$ randomized competitive algorithm if $1 \leq w_i \leq k$ for all $i$.

The duration of the exam is 3 hours. GOOD LUCK