On-line and Approximation Algorithms **Fall Semester, 2011/12**

Final Exam: February 8, 2012

Lecturer: Prof. Yossi Azar

Write short but full and accurate answers. Each question should start on a new page and each of its parts should not exceed a page. Solving 4 questions results in score 90. Two pages A4 (both sides each) are allowed.

1. Consider the following variant of ski rental problem. As usual it costs 1 to rent per time unit. If you buy the skis at time t then the price is $M + \alpha t$ for a given $\alpha > 0$.
   (a) Find the best competitive deterministic algorithm you can.
   (b) Show that no deterministic algorithm can achieve a better competitive ratio.
   (c) How would your answer change if the price at time t is $\max\{M - \alpha t, 0\}$ for a given $\alpha > 0$.

2. Consider a paging problem of pages of different sizes. Page i has size x_i which is an integer number between 1 and k. The memory can hold pages of total size k. The cost of bringing a page i to the memory is equal to its size. Once you remove part of a page P from the memory then P becomes unavailable in the memory and you would need to bring the whole page P at the next request for P.
   (a) Design a deterministic k competitive algorithm.
   (b) Assume that for all i we have $x_i = r$. Design a $\lfloor k/r \rfloor$ competitive deterministic algorithm.

3. Consider the on-line load balancing problem on m machines (m is even). Job i has a subset of machines S_i of size $m/2$ and two values w_i and p_i. Assigning job i is assigned to a machine $j \in S_i$ increases the load of j by w_i where assigning job i to a machine $j \notin S_i$ increases the load of j by p_i. The goal is to minimize the maximum load.
   (a) Design a $3 - \frac{2}{m}$ competitive algorithm.
   (b) Show a lower bound of 2 for any (even) m.

4. Consider an on-line load balancing on m unrelated machines located on a cycle. A job i has load $p_{ij} \geq 0$ if assigned to machine j. A job i arrives with an integer $1 \leq r_i \leq m$ and should be assigned to r_i consecutive machines on the cycle (one copy to each such machine).
   (a) Design an $O(\log m)$ competitive algorithm for minimizing the maximum load.
   (b) Consider the special case where $r_i = 2$ for all i. Moreover, for any i and j we have that p_{ij} is either 1 or m. Show an $\Omega(\log m)$ lower bound for any deterministic algorithm.

5. Consider admission control on a line $[0, n]$ of n unit edges. We are given a maximum value $\mu \geq 2$. Each request is an integer interval (a_i, b_i) of value v_i where $1 \leq v_i \leq \mu$. A solution is feasible if the accepted intervals are disjoint. The goal is to maximize the total value of the accepted request.
   (a) Design a randomized preemptive algorithm which is $O(\log n \cdot log\mu)$ competitive. Hint: think first on $\mu = 2$
   (b) Assume $\mu = n$. Show an $\Omega(\sqrt{n})$ lower bound for any deterministic preemptive algorithm. Hint: Use a long interval and short ones.

The duration of the exam is 3 hours. GOOD LUCK