Write short but full and accurate answers. Each question should start on a new page and each of its parts should not exceed a page. Solving 4 questions results in score 90. Two pages A4 (both sides each) are allowed.

1. Consider the following variant of ski rental problem. As usual it costs 1 to rent per time unit. If you buy the skis at time $t$ then the price is $M + \alpha t$ for a given $\alpha > 0$.
   (a) Find the best competitive deterministic algorithm you can.
   (b) Show that no deterministic algorithm can achieve a better competitive ratio.
   (c) How would your answer change if the price at time $t$ is $\max\{M - \alpha t, 0\}$ for a given $\alpha > 0$.

2. Consider a paging problem of pages of different sizes. Page $i$ has size $x_i$ which is an integer number between 1 and $k$. The memory can hold pages of total size $k$. The cost of bringing a page $i$ to the memory is equal to its size. Once you remove part of a page $P$ from the memory then $P$ becomes unavailable in the memory and you would need to bring the whole page $P$ at the next request for $P$.
   (a) Design a deterministic $k$ competitive algorithm.
   (b) Assume that for all $i$ we have $x_i = r$. Design a $\lfloor k/r \rfloor$ competitive deterministic algorithm.

3. Consider the on-line load balancing problem on $m$ machines ($m$ is even). Job $i$ has a subset of machines $S_i$ of size $m/2$ and two values $w_i$ and $p_i$. Assigning job $i$ is assigned to a machine $j \in S_i$ increases the load of $j$ by $w_i$ where assigning job $i$ to a machine $j \notin S_i$ increases the load of $j$ by $p_i$. The goal is to minimize the maximum load.
   (a) Design a $3 - 2/m$ competitive algorithm.
   (b) Show a lower bound of 2 for any (even) $m$.

4. Consider an on-line load balancing on $m$ unrelated machines located on a cycle. A job $i$ has load $p_{ij} \geq 0$ if assigned to machine $j$. A job $i$ arrives with an integer $1 \leq r_i \leq m$ and should be assigned to $r_i$ consecutive machines on the cycle (one copy to each such machine).
   (a) Design an $O(\log m)$ competitive algorithm for minimizing the maximum load.
   (b) Consider the special case where $r_i = 2$ for all $i$. Moreover, for any $i$ and $j$ we have that $p_{ij}$ is either 1 or $m$. Show an $\Omega(\log m)$ lower bound for any deterministic algorithm.

5. Consider admission control on a line $[0,n]$ of $n$ unit edges. We are given a maximum value $\mu \geq 2$. Each request is an integer interval $(a_i, b_i)$ of value $v_i$ where $1 \leq v_i \leq \mu$. A solution is feasible if the accepted intervals are disjoint. The goal is to maximize the total value of the accepted request.
   (a) Design a randomized preemptive algorithm which is $O(\log n \cdot \log\mu)$ competitive. Hint: think first on $\mu = 2$
   (b) Assume $\mu = n$. Show an $\Omega(\sqrt{n})$ lower bound for any deterministic preemptive algorithm. Hint: Use a long interval and short ones.

The duration of the exam is 3 hours. GOOD LUCK