Write short but full and accurate answers. Each question should start on a new page and each of its parts should not exceed a page. Solving 5 questions results in score 90. Two pages A4 (both sides each) are allowed.

1. Consider the ski rental problem (1 to rent, \( M \) to buy) with the following frequent skier program. After \((1 + \gamma)M\) times (for some fixed real number \( \gamma > 0 \), for example, \( \gamma = 0.7 \) or \( \gamma = 2.3 \)) you have rented the skis you get them for free.

   (a) Find the best deterministic strategy and its competitive ratio (as a function of \( \gamma \)).

   (b) Show a tight lower bound for any deterministic algorithm for the problem.

2. Consider a variant of the paging problem in which at each page fault the algorithm is allowed to change at most \( r \) pages in the memory for a cost of 1. Note that clearly \( r \leq k \) which is the size of the memory.

   (a) Find the best deterministic competitive algorithm for this variant (as a function of \( k \) and \( r \))

   Remark: note that also the optimal algorithm can change up to \( r \) pages in the memory for a cost of 1.

   (b) Design a matching lower bound for any deterministic algorithm for any \( k \) and \( r \). Recall that the online algorithm may replace up to \( r \) pages when there is a page fault at cost of 1.

   (c) Design \( O(\log k) \) randomized algorithm for \( r = 7 \). Hint: By how much at best the optimal cost can be improved here.

3. Consider the on-line load balancing problem on (even) \( m \) related machines. The speed of each of the first \( m/2 \) machines is 1 and the speed of each of the other \( m/2 \) machines is 100. The goal is to minimize the maximum load.

   (a) Show a deterministic 2.01 competitive algorithm.

   (b) Show a lower bound of 1.5 for any deterministic algorithm for any (even) \( m \geq 4 \) (not only for \( m = 4 \)).

4. We are given a connected graph \( G = (V, E) \). All edges have unit capacity. At round \( i \) we receive a request to connect \( s_i \) to \( t_i \) with one of two options. The first is on a path \( Q_i \) of bandwidth \( q_i \) the second is on a path \( R_i \) of bandwidth \( r_i \). We can choose at most one of the two options or reject both. Also \( r_i \leq q_i \leq \frac{1}{\log(2|V|+2)} \). Our goal is to maximize the total throughput while maintaining the capacity constraints. Design \( O(\log(|V|)) \) competitive algorithm.

   Remark: You need to state the algorithm exactly and prove its competitive ratio. Do not reprove the Lemmas we proved in class.

5. Consider admission control for the edge disjoint paths problem on a for a line of length \( 3n \) where the two end points of the line are stars (each is a center with additional \( k \) paths each of \( n \) edges). The goal is to maximize the number of accepted paths.

   (a) Design a deterministic preemptive algorithm which is \( O(\log n) \) competitive.

   (b) Design a randomized non-preemptive algorithm which is \( O(\log n) \) competitive.

6. Suppose you are in some point inside an island of an unknown planner shape. You can walk in any continues curve in the island. Your goal is to reach the seashore using a curve of minimum length. Once you reach the seashore (and only at that time) you know you are done. Show a constant competitive deterministic algorithm (you may assume that you are at least one unit of distance from the seashore and hence non additive constant is needed).

The duration of the exam is 3 hours. GOOD LUCK