Online Algorithms	28/12/11

Lecture Notes 9: Admission Control

Professor: Yossi Azar *Scribe:Roi Meron*

Introduction

In the following lecture we will cover the following topics:

- Admission Control - prove that the exponent algorithm is $O(\log(D))$ competitive.
- Admission Control on the line - lower bounds
- Interval Scheduling - the Classify and Randomly Select algorithm has an expected competitive ratio of $O(\log D)$

1 Admission Control

Reminder: In the Admission Control problem we are given a graph $G = (V, E)$, capacities over the edges, $c : E \to \mathbb{R}^+$ and a sequence of requests. A request is in the form (s_i, t_i, p_i, b_i) where:

- s_i - source
- t_i - target
- p_i - price/demand/bandwidth required
- b_i - benefit for servicing the request

Each request can be serviced or can be rejected and the goal is to maximize the benefit while maintaining a feasible solution: $\forall_e \sum_{i|e\in Q_i} p_i \leq c(e)$.

In the previous lecture we saw the *exponent* algorithm for this problem: "Accept a request i on a path Q_i if $\sum_{e\in Q_i} X_e \leq D$, else, reject."

Where:

$$X_e = (2D + 2)^{l_e} - 1$$

Theorem 1.1. *The exponent algorithm is feasible*

Proof. Assuming $p_i \leq \frac{1}{\log(2D+2)}$, overflow on edge e may occur only if $l_e > 1 - \frac{1}{\log 2D+2)}$ but then:

$$\begin{aligned} X_e &> (2D + 2)^{1-\frac{1}{\log(2D+2)}} - 1 \\ &= \frac{2D + 2}{2} - 1 \\ &= D \end{aligned}$$

But this implies that the algorithm would not have used this edge. □

In order to prove the competitive ratio, we are going to assume that:

- (*mandatory*) $p_i \leq \frac{c(e)}{\log(2D+2)}$ ($D \leq n-1$ is the maximal path length).
- $b_i = p_i$ - we try to maximize the throughput
- $c(e) = 1$

Theorem 1.2. *The exponent algorithm for Admission Control is $O(\log D)$ competitive.*

Proof. We will use the following lemmas.

Lemma 1.3. $\overbrace{2D \cdot \log(2D+2)}^{normalization} \sum_{i \in A} p_i \geq \sum_{e \in E} X_e$

Lemma 1.4. $\overset{\overset{normalization}{\downarrow}}{D} \cdot \sum_{i \in RA} p_i \leq \sum_{e \in E} X_e$ (RA are the requests we rejected and OPT accepted)

Lemmas 1.3, 1.4 prove the theorem:

$$\begin{aligned}
\sum_{i \in OPT} p_i &\leq \sum_{i \in A} p_i + \sum_{i \in RA} p_i \\
&\overset{Lemma 1.4}{\leq} \sum_{i \in A} p_i + \frac{1}{D} \sum_{e \in E} X_e \\
&\overset{Lemma 1.3}{\leq} \sum_{i \in A} p_i + \frac{2D \cdot \log(2D+2)}{D} \sum_{i \in A} p_i \\
&= \sum_{i \in A} p_i \cdot \underbrace{(1 + 2\log(2D+2))}_{\text{competitive ratio}} \\
&= O(\log D) \cdot \sum_{i \in A} p_i
\end{aligned}$$

□

Proof. **Lemma 1.3**
At time 0, $\forall_{e\in E}$: $X_e = 0$ and the gain is 0.
We will show that if the algorithm accepts the ith request, then,

$$2D \cdot \log(2D+2)p_i \geq \sum_{e\in E} \Delta X_e = \sum_{e\in Q_i} \Delta X_e$$

We will look at the change in the exponent function, $\sum\limits_{e\in Q_i} \Delta X_e$:

$$\begin{aligned}
\sum_{e\in Q_i} \Delta X_e &= \sum_{e\in Q_i} \overbrace{(2D+2)^{l_e+p_i}}^{\text{new weight}} \cancel{-1} - \overbrace{(2D+2)^{l_e}}^{\text{prev. weight}} \cancel{+1} \\
&= \sum_{e\in Q_i} (2D+2)^{l_e} \left[(2D+2)^{p_i} - 1\right] \\
&\overset{a=2^{\log_2 a}}{=} \sum_{e\in Q_i} (2D+2)^{l_e} (2^{\overbrace{\log(2D+2)p_i}^{\leq 1 \text{(light-weight edges)}}} - 1) \\
&\overset{0\leq x\leq 1 \Rightarrow 2^x - 1 \leq x}{\leq} \sum_{e\in Q_i} (2D+2)^{l_e} \log(2D+2)p_i \\
&= \log(2D+2)p_i \sum_{e\in Q_i} \overbrace{(2D+2)^{l_e} - 1}^{\leq D} + 1 \\
&\leq \log(2D+2)p_i \cdot (\underset{\text{accepting the request}}{D} + \underset{\text{path length}}{D}) \\
&= 2D \log(2D+2)p_i
\end{aligned}$$

□

Proof. **Lemma 1.4**
Q_i^* - The path that OPT uses for request i.
If the algorithm rejects request, specifically(because every path $> D$),

$$D < \sum_{e\in Q_i^*} X_e$$

Multiplying by p_i,

$$D \cdot p_i \leq p_i \sum_{e\in Q_i^*} X_e$$

Summing over all $i \in RA$

$$D \sum_{i\in RA} p_i \leq \sum_{i\in RA} p_i \cdot \sum_{e\in Q_i^*} X_e = \sum_{\substack{i\in RA \\ e\in Q_i^*}} X_e p_i$$

By replacing the order of summation:

$$= \sum_{e \in E} X_e \sum_{\substack{i | e \in Q_i^* \\ i \in RA}} p_i \leq \sum_{\substack{i \in RA \\ e \in Q_i^*}} X_e \cdot 1$$

where the last inequality follows since the weight that OPT puts on a single edge is ≤ 1.

□

Remarks:

1. *Variation*: A request arrives with a path Q_i. The algorithm can choose whether to accept or to reject. If $\sum_{e \in Q_i} X_e \leq D$, accept, else, reject. The proof remains the same.
2. The request does <u>not</u> arrive with a path. The proof is valid without assuming that the algorithm chooses the shortest path, but just a path of weight $\leq D$
3. if $p_i \leq b_i \leq \mu \cdot p_i$ (The benefit per cost unit is $\in [1, \mu]$), i.e. the ratio between a maximal benefit to a minimal benefit per bandwidth unit is $\leq \mu$, then, the competitive ratio is $O(\log D + \log \mu)$ using a more technical yet similar proof.

2 Lower Bounds

In the previous lecture we showed a lower bound of $\Omega(\log \mu)$ for $D = 1$.
We will now show a lower bound of $\Omega(\log D)$ for *any* D and see that the online algorithm is optimal(up to a constant ratio).

Theorem 2.1. *Any online algorithm for the Admission Control problem is at least* $\Omega(\log D)$*-competitive, "Even" in case of a line of length* D*, Even if Random, Even if Fractional*[1]*.*

disclaimer: A single line (a simple path) is not necessarily the simplest case, nor the simplest graph, however, proving it for the path case ensures us that every generic algorithm for the admission control problem cannot be better than $O(\log D)$-competitive.

Proof. The following sequence of requests comprises at most $\log(D)$ phases. In every phase we get many identical requests, each of small bandwidth:
Denote by X_i the gain of the online algorithm at phase i.
- At phase 0 the requests are $[0, D]$ and the total gain is X_0.
- At phase 1 the requests are $\left[0, \frac{D}{2}\right], \left[\frac{D}{2}, D\right]$ and the total gain is X_1.
- At phase i the requests are $\left[0, \frac{D}{2^i}\right], \ldots,$ and the gain is X_i.
The maximal phase is $\log D$, the requests are $[0, 1], [1, 2], \ldots, [D-1, D]$ and the gain is $X_{\log D}$.
The sequence may stop at any phase.

[1]Fractional algorithm - the algorithm can service even just a part of a request, or split it into several requests. Its benefit is proportional to the size it serviced.

In order to get a feasible solution, $\sum \frac{X_i}{2^i} \leq 1$, however, if we stop at phase j, then $OPT = 2^j$ (it takes only the requests in the last phase), so:

$$S_j = \frac{ON}{OPT} = \frac{\sum_{i=0}^{j} X_i}{2^j}$$

Summing over all possible S_j (all the possible phases where the sequence ends):

$$\begin{aligned}\sum_{j=0}^{\log D} S_j &= \sum_{j=0}^{\log D} \sum_{i=0}^{j} X_i \cdot 2^{-j} \\ &= \sum_{0 \leq i \leq j \leq \log D} X_i \cdot 2^{-j}\end{aligned}$$

replacing the order of summation:

$$= \sum_{i=0}^{\log D} X_i \sum_{j \geq i} 2^{-j}$$

by sum of geometric series:

$$\begin{aligned}&\leq \sum_{i=0}^{\log D} X_i \cdot 2 \cdot 2^{-i} \\ &= 2 \sum_{i=0}^{\log D} \frac{X_i}{2^i} \\ &\leq 2\end{aligned}$$

This implies that there exist j s.t. $S_j \leq \frac{2}{\log D+1}$.
For this j the sequence ends and we get a $\Omega(\log D)$-competitive ratio. □

Theorem 2.2. *If the requests have a demand(bandwidth) of $\frac{1}{k}$, then, any deterministic algorithm is $\Omega(D^{\frac{1}{k}})$-competitive, "Even" on a simple path.*
This theorem shows that the assumption $p_i \leq \frac{c(e)}{\log(2D+2)}$ is indeed mandatory.

Proof.

- $k = 1$:
  The sequence starts with a request of length D ($[0, D]$) which ON must accept, and blocks the network(because the capacity is 1), then, we give D requests of length 1 ($[0, 1], [1, 2] \dots$) that OPT accepts.

$$OPT = D, ON = 1 \Rightarrow \Omega(D)$$

- $k = 2$:
  Phase 1: 2 requests of length D ($[0, D]$) and demand $\frac{1}{2}$. ON accepts <u>at least</u> one.
  Phase 2: We give a prefix of the following sequence(two requests for every interval):

  $$[0, D^{\frac{1}{2}}], [D^{\frac{1}{2}}, 2D^{\frac{1}{2}}], \ldots [j \cdot D^{\frac{1}{2}}, (j+1)D^{\frac{1}{2}}] \ldots, [D - D^{\frac{1}{2}}, D]$$

  If ON rejects all of these requests, then OPT accepts all:

  $$OPT = D^{\frac{1}{2}}, ON = \frac{1}{2} \Rightarrow \frac{OPT}{ON} = \Omega(D^{\frac{1}{2}})$$

  Therefore, ON must accept a request at some point.
  Assume it accepts the jth:

  $$[j \cdot D^{\frac{1}{2}}, (j+1)D^{\frac{1}{2}}]$$

  At this point we continue by giving the following requests(two requests for every interval):

  $$[j \cdot D^{\frac{1}{2}}, j \cdot D^{\frac{1}{2}} + 1], [j \cdot D^{\frac{1}{2}} + 1, j \cdot D^{\frac{1}{2}} + 2], \ldots, [(j+1)D^{\frac{1}{2}} - 1, (j+1)D^{\frac{1}{2}}]$$

  OPT accepts all of them and we get: $ON = 1, OPT = D^{\frac{1}{2}} \Rightarrow \Omega(D^{\frac{1}{2}})$

- for any k:
  Each phase, we split the current interval into $D^{\frac{1}{k}}$ segments. We assume that ON picks one(otherwise, we are done), and on the next phase, we split that segment into $D^{\frac{1}{k}}$ segments, and so on. We can continue doing this process for k phases and on the last phase, ON blocks this interval with a single request and OPT gets a sequence of requests of length 1 and accepts them $\Rightarrow \Omega(D^{\frac{1}{k}})$.

□

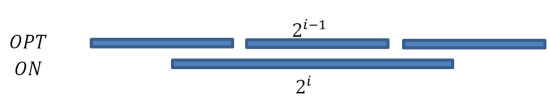
3 Interval Scheduling

We have an interval $[0, D]$, all request $[a, b](a < b \in \mathbb{N})$ have a demand=1 and the capacity is 1.
A feasible solution - disjoint paths (no overlapping requests).
A *deterministic* algorithm will be $\Omega(D)$-competitive: we will give requests $[0, D]$ until it accepts one, and the interval is blocked, and then the sequence continues with $[0, 1], [1, 2] \ldots$ and OPT will accept all.

Definition (bounded input)**.** *The input has only requests of length $2^{i-1} \leq l \leq 2^i$:*

Theorem 3.1. *For the bounded input case, the greedy algorithm ("accept when you can") is 3-competitive.*

Proof. for every segment ON accept, OPT can accept up to 3 segments instead. □



Classify and Randomly Select Algorithm

The algorithm:

- *Classify*: Split the input into $\log D$ sets, by length, i.e. the ith set are the requests of interval length 2^{i-1} to 2^i.
- *Randomly*: Choose $1 \leq i \leq \log D (i \in \mathbb{N})$ randomly and uniformly.
- *Select*: Accept only requests from the ith set.

Theorem 3.2. *The Classify and Randomly Select algorithm has an expected competitive ratio of $O(\log D)$.*

Proof. Denote by B_i the number of requests accepted by OPT in the ith set. Also denote by C&RS the online algorithm.
Using theorem 3.1,

$$\begin{aligned}
\mathbb{E}(\text{C\&RS}) &\geq \underbrace{\frac{1}{\log D}}_{\text{coin toss}} \left(\frac{1}{3} B_1 + \frac{1}{3} B_2 + \cdots + \frac{1}{3} B_{\log D} \right) \\
&= \frac{1}{3 \log D} \left(B_1 + B_2 + \cdots + B_{\log D} \right) \\
&= \frac{1}{3 \log D} OPT
\end{aligned}$$

□

Remark: In general, if we classify the input into k classes and in each one we have a C-competitive algorithm, then by using C&RS we get a $O(kC)$-competitive algorithm.