Online Algorithms 11/01/2012

Lecture Notes 11: Some More Scheduling

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Introduction

In this lecture we'll continue to discuss Admission Control on the line, but now each request's value equals its length. next, we'll get back to job scheduling - with a time factor added.

1 Admission Control: Value=Length

In this model, our goal is to maximize the overall value, where each request's value equal its size (in the previous model, each request's value was 1)

1.1 Deterministic & Non-Preemptive Algorithm

Theorem 1.1: *Greedy is n-competitive*
Proof.

- Greedy will accept at least a single request with size 1: $ON(\sigma) \geq 1$
- Opt will accept at most n: $OPT(\sigma) \leq n$

Theorem: *Every deterministic non-preemptive algorithm is at least n-competitive*
Proof.

- every algorithm will have to accept the first request with size 1 (otherwise he would be non-competitive)
- the following request will be of size n: OPT will accept that one.

1.2 Random & Non-Preemptive Algorithm

Theorem: *CRS (Classify & Randomly Select)+Greedy is $O(\log n)$-competitive*
Proof. in each class, if every accepted request is of size $[2^{i-1}, 2^i)$:

- $ON(\sigma)$ will accept a request with a size a: $a \geq 2^{i-1}$
- Opt can accept 2 additional requests with a size $\leq 2^i$: $OPT(\sigma) \leq 2 \cdot 2^i + a$
- $\frac{OPT(\sigma)}{ON(\sigma)} \leq \frac{2 \cdot 2^i + a}{a} \leq 1 + \frac{2 \cdot 2^i}{a} \leq 5$
- there are $\log n$ classes, so overall we get $5 \cdot \log n = O(\log n)$ competitive algorithm.

The following theorem shows that one cannot achieve a better competitive algorithm.

Theorem: *Any non-preemptive randomized algorithm is* $\Omega(logn)$ *competitive*

Proof. each time $ON(\sigma)$ accepts a request, it misses a request sized n. So, as we've seen in One Way Trading:

- in this case, stock price range (here, request size range) in $[1, n]$. i.e, $\mu = n$
- we already know that the lower bound for One Way Trading is $log\mu \longrightarrow \Omega(logn)$

1.3 Deterministic Algorithm with Preemption Allowed

"Anti-Reasonable" Algorithm: for any request R, accept if both intersected requests (from both sides) is **at least** 2(or perhaps a better constant: **g**)-times shorter.

Figure 1: request no. 4 accepted because 1,2,3 are more than 2-times shorter. thus, they are preempted

Theorem: *"Anti-Reasonable" is constant-competitive*

Proof. We can construct the same request-tree, as we did in the case where value=1. When $ON(\sigma)$ accepts a request with a size $|x|$:

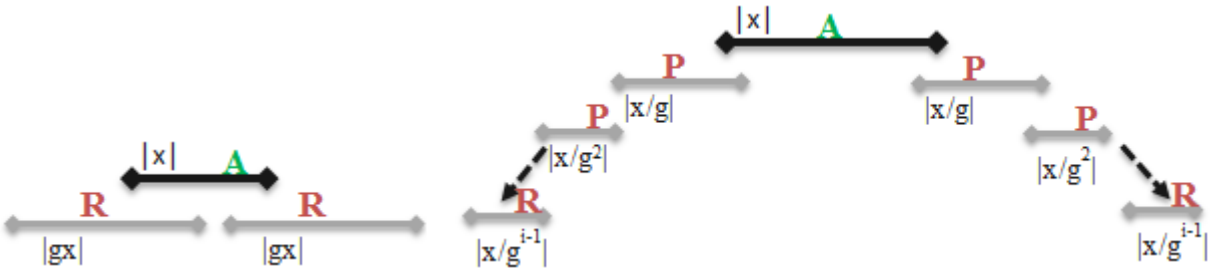


Figure 2: in both cases: $ON(\sigma)$ lost $2|gx|$

- Thus, $ON(\sigma) \geq x$ and $OPT(\sigma) \leq x(1 + 2g)$
- If we set $g = \frac{1}{g} + 1$ then according to the golden ratio:$g = \frac{1+\sqrt{5}}{2}$
- Following that the competitive ratio equals $1 + 2g = 2 + \sqrt{5} = 4.236...$

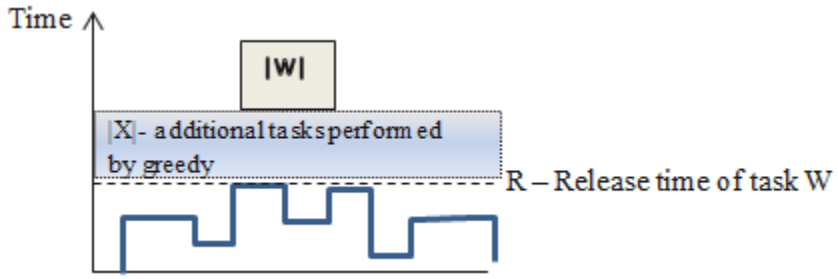
2 Task Scheduling: with release & completion time

in this case every task has, in addition to its size property, a release time. A machine can process only one task at a given time, and our goal here is to process all tasks as fast as possible.

2.1 Reduction to Batch

First, we'll discuss the case of identical machines.
we will use the **greedy algorithm**: apply a task on the first idle machine.
remark: greedy must be at least $2 - \frac{1}{m}$ competitive, because Load Balancing for identical machine is a special case here (release time of every task is in $[0, +\epsilon]$).

Theorem: *Greedy is 2-competitive (for the identical machine model)*
Proof.



- $OPT(\sigma) \geq |x|$, $OPT(\sigma) \geq r + |w|$ - due to volume conservation
- $ON(\sigma) = r + |x| + |w| \leq 2OPT(\sigma)x$

Reduction to batch: we can wait certain time period, and then process all tasks that have arrived as offline $OPT(\sigma)$ would.

- Initialization: $t_0 = 0$, $k = 0$
- collect all tasks with release time $\leq t_k$, and yet to be processed
- solve the offline problem (or use a polynomial time α-approximation algorithm).
- Let Δt_k the time to process these jobs,
- Let $t_{k+1} = t_k + \Delta t_k$

Theorem: Reduction to Batch is 2α-competitive for any machine model.
Proof.

- assume that the last task arrived between t_{k-1} to t_k
- **recall** Δt_k: processing time of tasks arrived in t_{k-1} to t_k
- $OPT(\sigma) \geq t_{k-1} + \frac{\Delta t_k}{\alpha}$
- $OPT(\sigma) \geq t_{k-2} + \frac{\Delta t_{k-1}}{\alpha} \geq \frac{\Delta t_{k-1}}{\alpha}$ (processing of tasks arrived in $[t_{k-2}, t_{k-1}]$)
- $ON(\sigma) = t_{k-1} + \Delta t_{k-1} + \Delta t_{k+} \leq \alpha OPT(\sigma) + \alpha OPT(\sigma) = 2\alpha OPT(\sigma)$

2.2 Minimum Flow Time

- **define** *Flow Time* of task i: $f_i \equiv c_i - r_i$, where c_i is the completion time of task i.
- we can use **preemption**: stop processing a task and resume later on, or **migration**: transfer a task to another machine.
- Now, our goal is to minimize the overall flow time: $Min \sum_i f_i$

At first, we assume m=1 and that all tasks are given at time $t_0 = 0$.

- **SPT(Shortest Processing Time First) Algorithm**: assign tasks by shortest processing time.

Theorem: SPT is optimal (if all tasks are given at time 0)
Proof.

if Preemption is not allowed: A solution is an order of the jobs with no "holes".

- assuming OPT did not order the tasks by size, i.e there are 2 tasks x, y consecutive in its order, where x is processed before y but $|x| > |y|$.
- if we process y before x:
  - completion time of any other request does not change (x and y are consecutive)
  - $\max\{c_x, c_y\}$ does not change but $\min\{c_x, c_y\}$ decreases which is a contradiction to the fact that this Algorithm is optimal.

if Preemption is allowed:

- Given a preemptive solution create non-preemptive solution according to the order of jobs completion. This can only reduce the flow time of each job (proof by induction) which is a contradiction to the fact that this Algorithm is optimal.

Now lets relate to a more general case- where r_i is not fixed:

- **SRPT(Shortest Remaining Processing Time First) Algorithm**: assign tasks by their remaining shortest processing time.

Theorem: SRPT is 1-competitive
Proof. Assuming there is an optimal scheduling that is not SRPT:

- let A and B be tasks that are not assigned SRPT.
- now, we switch every process-slot of A with process-slot of B and vice versa.
  - Other tasks remain in tact.
  - the task that finishes last,will have the same completion time
  - but for the other one, it's completion time is decreased which is a contradiction to the fact that this Algorithm is optimal.