Algorithmic Methods	27/12/2010

Lecture Notes 11: Local Ratio Cont.

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1 Introduction

This lecture starts by reviewing the main points and differences between the local ratio algorithms studied previously. We then present the NP-Hard problem *Steiner tree* and show a 2-approximation algorithm for it. A different approximation algorithm is following, this time using the local ratio technique. This algorithm generalizes naturally to the *Generalized Steiner Forest* problem, introduced at the end of this lecture.

2 *VC* and *IS* Local Ratio Algorithms Recap

Algorithm Step	ISB-2	ISD-2	IS-1	VC-2
Throw useless input. If the input is trivial, return a solution with the required approximation	$v_i \leq 0$	$v_i \leq 0$	$v_i \leq 0$	$-$
Break weights into degenerated set and general (smaller) set	Take the first ending interval. All intersecting intervals will be of value $b_i v/(1-b)$	Take the first ending copy. All intersecting intervals will be of value v	Take the first ending interval. All intersecting intervals will have value v	Take the less costly vertex incident to an arbitrary edge
Recursive solution	First interval removed	First living copy removed	First interval removed	There's at least one more zero-valued vertex.
Fix the solution to match the original problem, keeping the approximation ratio	Add first interval if possible (i.e., total bandwidth of intersecting intervals $\leq 1-b$)	Add interval copy if possible (if no other interval intersects and no other copy was taken)	Add first interval if possible (recall its value is 0)	No fixing is required. Accepted solution is the recursive solution
Trivial input	$\phi \to \phi$	$\phi \to \phi$	$\phi \to 0$	zero-valued vertices form VC
Proof:(cost) $OPT \geq V$, $ANYFIXED \leq \alpha V$ (α is the approximation ratio)	$OPT \leq 2V$, $ANYFIXED \geq V$	$OPT \leq 2V$, $ANYFIXED \geq V$	$OPT = V$, $ANYFIXED = V$	$OPT \geq V_1$, $ANYFIXED \leq 2V_1$

Theorem. If there exists a solution S such that

$$w_1(S) \leq \alpha OPT(I, w_1)$$

$$w_2(S) \leq \alpha OPT(I, w_2)$$

Then

$$w(S) \leq \alpha OPT(I, w_1 + w_2).$$

3 Steiner Tree

Definition. Let $G = (V, E)$ a connected, edge-weighted graph and $S \subseteq V$. A *Steiner Tree* with respect to G, S is a minimum-weight subgraph of G spanning all the vertices in S.

3.1 2-Approximation Algorithm

- Reconstruct the edges of G, such that the distance between $u, v \in S$ is the weight of the shortest path connecting them. Denote the subgraph of G induced by the vertices of S by G'.
- Find MST T on G'.
- Translate T back into a solution to the original graph G by converting edges back into paths and throwing spare edges.

Claim 1. The algorithm above is a 2-approximation algorithm.

Note. Any spanning tree for G' is a feasible solution for the Steiner Tree problem.

Proof of Claim 1. Consider a Steiner tree of cost OPT. Double the edges of OPT to obtain an Euler graph. Thus there is a cycle (traverses once in each edge) of weight $2OPT$. Shortcutting this cycle (by skipping vertices that were already visited or not belonging to S) will yield a new cycle in G' comprised of all vertices of S. Clearly, this operation does not increase the cost of the solution, as the shortcutting edges represent the weight of the shortest path between each pair of vertices in S. Finally we have a spanning tree of G' of cost $2OPT$ at most. This is an upper bound for the MST T of G' returned by our algorithm. Therefore a 2-approximation for the problem is achieved.

Note. There exist algorithms with a better approximation ratio in the general case, and particularly in an Euclidean graphs.

3.2 Approximating Steiner Tree by Local Ratio Algorithm

Let $G = (V, E)$, $S \subseteq V$. The following algorithm uses the local ratio technique to approximate Steiner tree problem by ratio 2.

Denote:

- $\varepsilon_1 = min\{c(u, v) | u \in S \text{ xor } v \in S\}$
- $\varepsilon_2 = min\{\frac{1}{2}c(u, v) | u \in S \text{ and } v \in S\}$
- $\varepsilon = min\{\varepsilon_1, \varepsilon_2\}$

Algorithm Steiner-Tree.

1. If $|S| = 1$ return the vertex in S and no edges.
2. For each $e = (u, v) \in S$, define $w_2(e) = \varepsilon$ if $|\{u, v\} \cap S| = 1$, or $w_2(e) = 2\varepsilon$ if $|\{u, v\} \cap S| = 2$.
   For all other edges $w_2(e) = 0$.
   Define $w(e) = w(e) - w_2(e)$.
3. If $w(e) = 0$ contract e, treating the new vertex as a vertex in S if any of its contracted vertices were in S. The edges of the new vertex will be the union of the edges of the contracted vertices (in case of double edges, use the cheaper one for the new vertex).

4. Solve recursively.

5. Expand the contracted vertices and add the edge connecting them with zero weight.

6. Leave only necessary edges in the graph (i.e., no cycle and all the leaves should be vertices in S).

Claim 2. Any fixed solution with degenerated weights (i.e, with respect to w_2) is a 2-approximation to OPT, namely for $|S| \geq 2$

- $OPT \geq |S| \cdot \varepsilon$
- $ANYFIXED \leq 2\varepsilon(|S| - 1)$

Proof of Claim 2. We start by showing that $OPT \geq |S| \cdot \varepsilon$. By induction on $|S|$. If $|S| = 2$ the minimal cost of the spanning tree would be 2ε, whether the vertices are connected directly or not. For a larger S, we have

$$OPT_S \geq OPT_{S-1} + \varepsilon \geq (|S| - 1) \cdot \varepsilon + \varepsilon = |S| \cdot \varepsilon.$$

This is due to the fact that the minimal cost of adding a vertex to the set S is ε. The second inequality is due to the induction hypothesis.

We now show that $ANYFIXED \leq 2\varepsilon(|S|-1)$. By induction on $|S|$. If $|S| = 2$, the fixed solution is exactly $2\varepsilon = 2\varepsilon(2 - 1)$. For larger S, consider the given solution. Traverse from a vertex in S until reaching another vertex in S or a vertex with degree at least 2. Fold the created path to remain with a tree containing $|S| - 1$ leaves in S. By induction,

$$A(|S|) \leq A(|S| - 1) + 2\varepsilon \leq 2(|S| - 2) \cdot \varepsilon + 2\varepsilon = 2(|S| - 1) \cdot \varepsilon.$$

4 Generalized Steiner Forest

Let $G = (V, E)$ undirected weighted graph, $\{(s_i, t_i)\}_{i=1}^{k}$ pairs of vertices in G. A feasible solution to the problem would be a subgraph of G (possibly unconnected) connecting s_i and t_i for all $i \in [k]$. The objective is to find a feasible subgraph which is *minimal*.

Two immediate solutions can be easily proved to be very costly:

- Take all the shortest paths s_i to t_i. This can be as bad as $|S| \cdot OPT$.
- Take the Steiner tree over all the vertices s_i, t_i. This can be unlimited worse than OPT.

Note. Steiner tree is a special case of this problem by taking all pairs of vertices in $|S|$ (there are $\binom{|S|}{2}$ such pairs).

Note. An equivalent presentation of the problem is as follows: Given n sets of vertices $S_1, S_2, ..., S_n$, find a minimal forest such that for all $u, v \in S_i$ there exists a path from u to v in the forest.
Proof of the Note. Consider the alternative presentation. For all i and for each $u, v \in S_i$, construct a pair (u, v). This gives an equivalent problem with the original presentation. For the other direction, simply construct a set of two vertices for each pair of the original presentation.