Algorithmic Methods	20.12.2010

Lecture Notes 10: Local Ratio

Professor: Yossi Azar *Scribe: Inna Kalp*

1 Introduction

In this Lecture we will introduce the Local Ratio Technique. We will show applications of this method to the following problems:

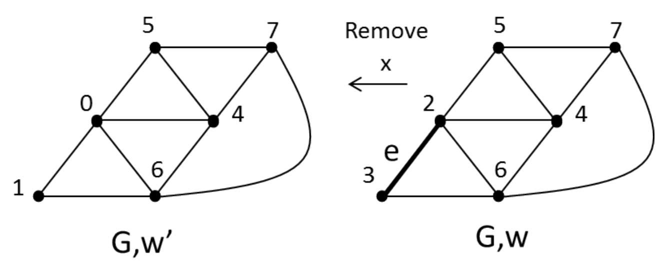
- Vertex cover
- Interval scheduling
- Interval scheduling with deadlines
- Interval scheduling with intervals of various widths.

2 Vertex Cover

Introducing the problem:
Input: Graph G = (V,E), weight function $w : V \to R^+$
Feasible solution: $S \subseteq V$ s.t $\forall e = (v, u) \in E$. $u \in S$ or $v \in S$
Goal Function: Minimize $\sum_{v \in S} w(v)$.
Claim 1: The value of the solution is 0 $\Longleftrightarrow$the group of all the vertexes with weight 0 is a valid cover $\Longleftrightarrow$$\forall e = (u, v) \in E$ $u \in S$ or $v \in S$
Proof: follows by definition.

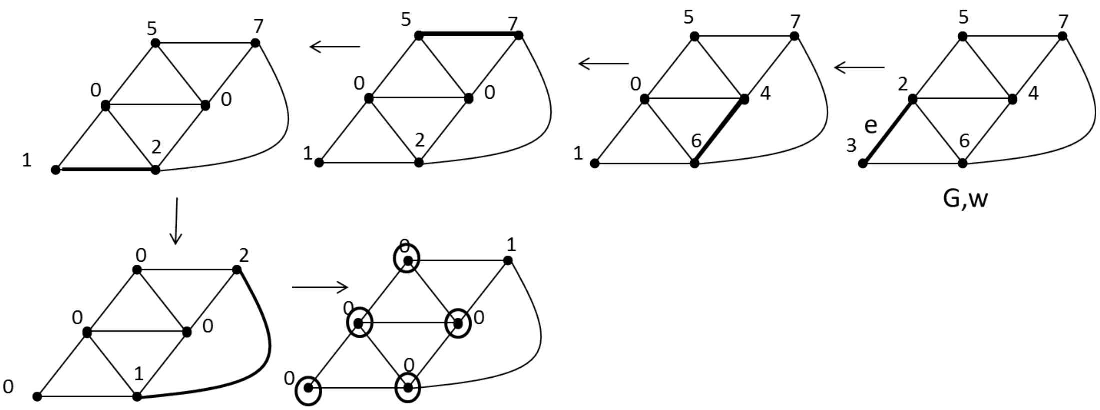
2.1 First Solution:

Given input: G = (V,E), w(v)$\rightarrow$ R^+, choose an edge e=(u,v)$\in$ E and subtract $X = min\{w(u), w(v)\}$ from w(v) and w(u). The result is graph G with a new weight function w'.
Figure 1: An example of G with weights w & w':



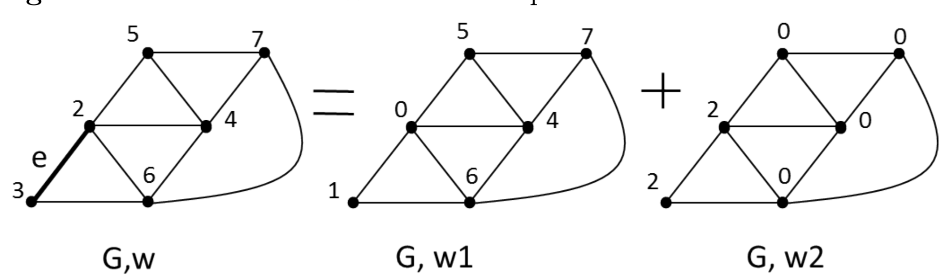
Notice that:

(1) OPT(G,w')≤OPT(G,w)-X
(2) A(G,w)≤A(G,w') + 2X
Explanations:
(1) if S is a feasible solution to (G,w) with value V , it will also be a feasible solution to (G,w') with value at most V-X.
(2) if S is a feasible solution to (G,w') with value V , it will also be a feasible solution to (G,w) with value at most V +2X (if both v and u are in S, we add 2X).
Claim 2: We can achieve a 2-approximation algorithm by recursion on (G,w'):
The Algorithm: choose an edge e$\in E$ and generate (G,w'). Continue until the weight of at least one of the vertexes of each edge is 0.
Proof: By induction: (on the number of edges (u,v) s.t w(u) >0 and w(v) > 0)
Base case: At least one of the vertexes of each edge is 0 $\rightarrow$ the value of the solution is 0 (we will take all the vertexes with weight 0). The solution is optimal, and therefore 2-approximate.
Induction Step: $A(G,w') \leq_{(2)} A(G,w')+2x \leq_{induction} 2OPT(G,w')+2x = 2(OPT(G,w')+x) \leq_{(1)} 2OPT(G,w)$
Figure 2: An Example of the recursive algorithm:



2.2 An alternative approach:

We divided the weight function w into 2 functions: w_1:V$\rightarrow$ $R^+$and w_2:V$\rightarrow$ R^+ s.t w = w_1+w_2.
Figure 3: The division of W in our example:

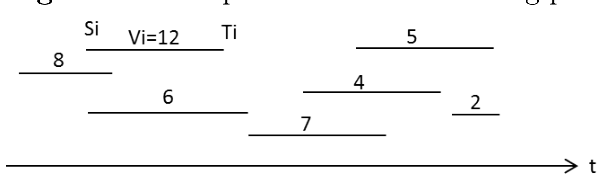


The Local Ratio Theorem: If solution S is α-approximate for (G,w_1) and (G,w_2), S is also $\alpha - approximate$ for (G,w)

Proof: $w(S) = w_1(S) + w_2(S) \leq \alpha OPT(G, w_1) + \alpha OPT(G, w_2) = \alpha(OPT(G, w_1) + OPT(G, w_2)) \leq \alpha OPT(G, w_1 + w_2) = \alpha OPT(G, w)$
The local Ratio Theorem applied to the Vertex Cover:
The algorithm: If a zero-cost solution can be found, return one. Otherwise, decompose w into w_1&w_2, and solve the problem recursively, using w_1 as the weight function in the recursive call.
Claim 3: The algorithm is 2-approximate.
Proof: By Induction: (on the number of edges (u,v) s.t w(u) $>$0 and w(v) $>$ 0)
Base case: The algorithm returns a VC of zero cost, which is optimal.
Inductive step: consider the solution returned by the recursive call. In (G,w1) there is one edge (u,v) less s.t w(u) $>$0 and w(v) $>$ 0, by the inductive hypothesis it is 2-approximate with respect to w_1. We claim that it is also 2-approximate with respect to w_2 . In fact, every feasible solution is 2-approximate with respect to w_2: OPT(G,w_2)$\geq$ x and Alg(G,w_2) $= \sum_{v \in S} w_2(v) \leq \sum_{v \in V} w_2(v) = 2x$. (G,$w_1$) and (G,$w_2$) is 2-approximate$\rightarrow$ (G,w) is 2-approximate.

3 Interval Scheduling

Introducing the problem:
Input: A set of intervals: $I_i = (s_i.t_i, v_i)$ $s_i \leq t_i$ v_i- the value of interval i
Feasible solution: subset S of non-conflicting intervals.
Goal Function: maximize the sum of values of S.
Figure 4: Example of Interval scheduling problem:



3.1 Simple Case: $\forall i.v_i = 1$:

The Goal is to maximize the number of intervals in S.
The Algorithm: Choose the interval that ends first, and remove all of its conflicting intervals. Repeat until there are no more intervals.
Claim 4: The Algorithm gives an optimal solution.
Proof: The chosen interval is better than any other interval, because it can be replaced by any of the other intervals we removed.

3.2 The General Case $\forall i.v_i \geq 0$:

3.2.1 Solution using dynamic programming:

Sort the intervals by the ending time(t_i):$I_1 \leq I_2 ... \leq I_n$
definition: F(I) = the maximum gain from intervals ending until the ending if the i-th interval ; F(0) = 0.

definition: J_i= the interval with maximum index that ends before the beginning of interval i.
$F(I) = \{F(i-1), v(i) + F(j_i)\}$

3.2.2 Solution using Local Ratio technique:

1. If there are interval s.t $v_i < 0$ - remove them. If the input is $\phi - return\ \phi$.
2. The weight function w_1: Select the interval that that ends first - $I_1$with value v and remove v from all of $I_1's$ conflicting intervals. (The weight function w_2 =w-w_1). Remove I_1.
3. Solve recursively.
4. Take the recursive solution and add $I_1$to it if possible (= if interval I_1doesn't conflict with the other intervals in the solution).

Claim 5: The algorithm returns an optimal solution.
Proof: By induction (on the number of intervals):
Base case: If the input is $\phi - return\ \phi$- optimal solution.
Inductive step: First, notice that the all the intervals with value > 0 are conflicting with each other, and OPT = v. Consider the solution S returned by the recursive call. In w_1 there is at least one interval less than in w. By the inductive hypothesis S is optimal with respect to w1.
If S contains an interval conflicting with I_1: There is an interval conflicting with I_1in the solution→ the value of the solution in Alg(w_2) = $v \rightarrow optimal$.
If S doesn't contain an interval conflicting with I_1we add it to the solution: $Alg(w_2) = v \rightarrow optimal$. S is optimal with respect to $w_1$and $w_2 \rightarrow S$ is optimal with respect to w = $w_1 + w_2$.

4 Interval Scheduling with deadlines

Introducing the problem:
Input: A set of intervals: $I_i = (s_{i.}t_i, d_i, v_i)$: interval i can be placed in the range $[s_i, t_i]$, the interval's length is d_i, and its value - v_i.
Feasible solution: Subset S of non-conflicting intervals, and the starting location for each selected interval in its range.
Goal Function: Maximize the sum of values of S.
Remark: The problem is NP-Complete (can be proved by reduction to Bin Packing problem).
Assumption: s_i, t_i, d_i are Integer.
The technique: Let's open each interval to a set of all of it's possible locations within the range $[s_i, t_i]$, and apply the algorithm described in section 3.2.2.
The Algorithm:

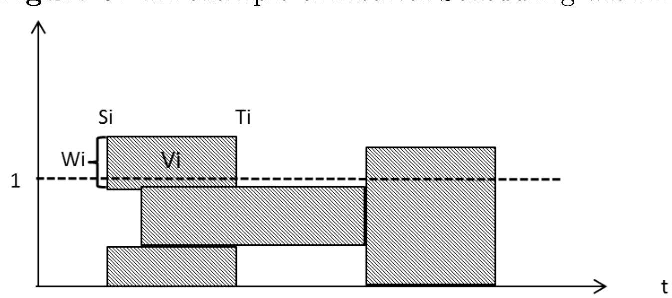
1. If there are interval s.t $v_{ij} < 0$ - remove them. If the input is $\phi - return\ \phi$.

2. The weight function w_1: Select the interval that that ends first - $I_1$with value v and remove v from: (1) all of $I_1's$ conflicting intervals. (2) all the copies of I_1. (the weight function w_2 =w-w_1). Remove I_1.
3. Solve recursively.
4. Take the recursive solution and add $I_1$to it if possible (= if interval I_1doesn't conflict with the other intervals in the solution, and there isn't any copy of I_1 in the solution).

Claim 6: The algorithm is 2-approximate.
Claim 7: Optimal solution for $w_2 \leq 2v$.
Proof (of Claim 7): We take v from all the intervals conflicting with I_1, and v from a copy of I_1.We get 2v.
Claim 8: The value of every maximal solution with respect to $I_1 \geq v$.
Proof (of Claim 8): By definition: If we take a copy of I_1-we get v. If we take a copy of an interval conflicting with I_1-we get v. Otherwise, we can add I_1 to the solution - we get v.
Corollary 1: From claims 7 and 8 we get that any feasible solution to w_2 is 2-approximate.
Proof (of Claim 6): By induction (on the number of intervals):
Base case: If the input is $\phi - return\ \phi$- optimal solution (and 2-approximate).
Inductive step: Consider the solution S returned by the recursive call. In w_1 there is at least one interval less than in w. By the inductive hypothesis S is optimal with respect to w1. We have also shown (in corollary 1) that any feasible solution to w_2 is 2-approximate. S is optimal with respect to $w_1$and $w_2 \rightarrow S$ is optimal with respect to w $= w_1 + w_2$.

5 Interval Scheduling with intervals of various widths:

Introducing the problem:
Input :A set of intervals: $I_i = (s_{i.}t_i, w_i, v_i)$: s_i - the start of the interval, t_i- the end of the interval, w_i,- the interval's width, v_i- the interval's value.
Feasible solution: Subset S of intervals, s.t at any given time, the sum of the widths of the intervals is at most 1.
Goal Function: Maximize the sum of values of S.
Figure 5: An example of Interval Scheduling with intervals of various widths:



The Algorithm:
<u>Phase 1:</u> Separate the input into 2 groups: $I_1 = \{i|w_i > \frac{1}{2}\}$; $I_2 = \{i|w_i \leq \frac{1}{2}\}$
<u>Phase 2:</u> Solve each group separately:

Solution to I_1 :_since $w_i > \frac{1}{2}$, at each given moment, there can be placed at most 1 interval. This is the regular Interval Scheduling problem- we have shown how to solve this problem optimally.
Solution to I_2 :

1. If there is an interval s.t $v_i < 0$ - remove item. If the input is $\phi - return\ \phi$.
2. The weight function w_1: Select the interval that that ends first - $I_1$with value v and weight w, and remove $\frac{v*w_i}{1-w}$ from all of $I_1's$ conflicting intervals. (w_i is the weight of the conflicting interval). The weight function w_2 =w-w_1. Remove I_1.
3. Solve recursively.
4. Take the recursive solution and add $I_1$to it if possible (= if interval I_1adding interval I_1doesn't result in a time t s.t the sum of the weights at that time is over 1).

Claim 9: Optimal solution for $I_2, w_2$$\leq 2v$.
Proof : $OPT \leq max\{v\frac{\sum w_i}{1-w}, v + v\frac{\sum w_i}{1-w}\} \leq max\{v\frac{1}{1-w}, v + v\frac{1-w}{1-w}\} = max\{2v, 2v\} = 2v.$
Explanation: The first option - OPT$\leq v\frac{\sum w_i}{1-w}$ represents the case where we didn't add $I_1$to the solution. In this case $\sum w_i \leq$ 1since it's a valid solution, and since $w_i > 1/2$ we get OPT$\leq 2v$. The second option(we add $I_1$to the solution): OPT$\leq v + v\frac{\sum w_i}{1-w}$- v for $I_1$and $v\frac{\sum w_i}{1-w}$ for all of $I_1's$ conflicting intervals. We can add I_1,so $\sum w_i + w \leq 1 \rightarrow \sum w_i \leq 1 - w$. We get OPT$\leq 2v$.
Claim 10: the value of every maximal solution with respect to $I_1, w_2$$\geq v$.
Proof: Alg$\geq min\{v, v\frac{\sum w_i}{1-w}\} \geq nim\{v, v\frac{1-w}{1-w}\} = min\{v, v\} = v.$
Explanation: The first option - Alg$\geq v$ represents the case where we add $I_1$to the solution- the solution value is at least v. . The second option(we can't add $I_1$to the solution): OPT$\geq v\frac{\sum w_i}{1-w}$- We can't add I_1,so $\sum w_i + w > 1 \rightarrow \sum w_i > 1 - w$. We get Alg$\geq v$.
Corollary 2: From claims 9 and 10 we get that any feasible solution to w_2 is 2-approximate.
Claim 11: The solution to $I_2$is 2-approximate.
Proof: The proof of claim 6 (the algorithm for interval scheduling with deadlines) also applies here.
Phase 3: We solve the problem for $I_1$and $I_2$and take the best solution.
Claim 12: The algorithm is 3-approximate.
Proof: OPT$\leq OPT(I_1) + OPT(I_2)$. If OPT(I_1)$\geq \frac{1}{3}OPT$:Alg(I_1) returns an optimal solution, so we get :Alg(I_1) $= OPT(I_1) \geq \frac{1}{3}OPT$. We get a 3-approximation. Otherwise, OPT(I_2)$\geq \frac{2}{3}OPT$. We got a 2-approximatiom to I_2, So We Get Alg(I_2) $\geq \frac{1}{2}$OPT(I_2) $\geq \frac{1}{2} * \frac{2}{3}OPT = \frac{1}{3}OPT$. In both cases we get 3-approximation.