Tutorial on Non-Deterministic Semantics
Part III: More Advanced Topics

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What this tutorial is about

Non-deterministic Semantics (Matrices):
Incorporating the notion of "non-deterministic computations" from automata and computability theory into logical truth-tables. We would like to show: Non-deterministic semantics is a natural and useful paradigm.
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Incorporating the notion of "non-deterministic computations" from automata and computability theory into logical truth-tables.

We would like to show:
Non-deterministic semantics is a natural and useful paradigm.
Already covered topics

- Basic definitions and properties of Nmatrices.
- Application: canonical Gentzen-type systems.
- Application: semantics and sequent calculi for Logics of Formal (In)consistency
Overview of Part III - More Advanced Topics

1. Constructive Canonical Systems
2. FOL Defs
3. FO C-systems
4. Canonical Systems with Quantifiers
Characterization of constructive connectives

- Extending the notion of canonical systems to the framework of single-conclusioned Gentzen-type calculi.
- Semantics: a combination of non-deterministic semantics with Kripke-style frames
Characterization of constructive connectives

- Extending the notion of canonical systems to the framework of *single-conclusioned* Gentzen-type calculi.
- Semantics: a combination of non-deterministic semantics with Kripke-style frames
- **Application**: constructive connectives can be characterized proof-theoretically by a set of canonical rules in *single-conclusion* canonical systems.
Reminder: what is a multiple-conclusioned canonical rule?

Stage 1.
\[ \Gamma, \psi, \varphi \Rightarrow \Delta \]
\[ \Gamma, \psi \land \varphi \Rightarrow \Delta \]
\[ \Gamma \Rightarrow \Delta, \psi \]
\[ \Gamma \Rightarrow \Delta, \varphi \]

Stage 2.
\[ \psi, \varphi \Rightarrow \]
\[ \psi \land \varphi \Rightarrow \]
\[ \Rightarrow \psi \]
\[ \Rightarrow \varphi \]
\[ \Rightarrow \psi \land \varphi \]

Stage 3.
\[ \{ p_1, p_2 \Rightarrow \}/ p_1 \land p_2 \Rightarrow \]
\[ \Rightarrow p_1 ; \Rightarrow p_2 \}/ \Rightarrow p_1 \land p_2 \]
What is a single-conclusioned canonical rule?

**Stage 1.**

\[
\begin{align*}
\Gamma, \psi, \varphi & \Rightarrow \theta \\
\Gamma & \Rightarrow \psi \\
\Gamma & \Rightarrow \varphi
\end{align*}
\]

\[
\Gamma, \psi \land \varphi \Rightarrow \theta
\]

\[
\Gamma \Rightarrow \psi \land \varphi
\]

**Stage 2.**

\[
\psi, \varphi \Rightarrow
\]

\[
\Rightarrow \psi \Rightarrow \varphi
\]

\[
\Rightarrow \psi \land \varphi
\]

**Stage 3.**

\[
\{ p_1, p_2 \Rightarrow \}/ p_1 \land p_2 \Rightarrow
\]

\[
\Rightarrow p_1 ; \Rightarrow p_2 \}/ \Rightarrow p_1 \land p_2
\]
Example 1

Implication rules:

\[ \{ p_1 \Rightarrow p_2 \} / \Rightarrow p_1 \supset p_2 \quad \{ \Rightarrow p_1 ; p_2 \Rightarrow \} / p_1 \supset p_2 \Rightarrow \]

Their applications:

\[ \Gamma, \psi \Rightarrow \varphi \quad \Gamma \Rightarrow \psi \quad \Gamma, \varphi \Rightarrow \theta \]
\[ \Gamma \Rightarrow \psi \supset \varphi \quad \Gamma, \psi \supset \varphi \Rightarrow \theta \]
Example 2

Semi-implication rules (Gurevich):

\[
\{ \Rightarrow p_1 ; p_2 \Rightarrow \} / p_1 \bowtie p_2 \Rightarrow \quad \{ \Rightarrow p_2 \} / \Rightarrow p_1 \bowtie p_2
\]

Their applications:

\[
\frac{\Gamma \Rightarrow \psi}{\Gamma, \psi \bowtie \varphi \Rightarrow \theta} \quad \frac{\Gamma \Rightarrow \varphi}{\Gamma \Rightarrow \psi \bowtie \varphi}
\]

\[
\frac{\Gamma, \varphi \Rightarrow \theta}{\Gamma \Rightarrow \psi \bowtie \varphi}
\]
A canonical single-conclusioned calculus \( G \) is **coherent** if for every pair of rules \( \Theta_1/ \Rightarrow \diamond (p_1, ..., p_n) \) and \( \Theta_2/ \diamond (p_1, ..., p_n) \Rightarrow \), the set of clauses \( \Theta_1 \cup \Theta_2 \) is classically unsatisfiable (and so inconsistent, i.e., the empty sequent can be derived from it using only cuts).

Examples of coherent calculi:

\[
\begin{align*}
\{p_1 \Rightarrow p_2\} & / \Rightarrow p_1 \supset p_2 \quad \{\Rightarrow p_1 ; p_2 \Rightarrow\} & / p_1 \supset p_2 \Rightarrow \\
\{\Rightarrow p_1 ; p_2 \Rightarrow\} & / p_1 \rightsquigarrow p_2 \Rightarrow \quad \{\Rightarrow p_2\} & / \Rightarrow p_1 \rightsquigarrow p_2
\end{align*}
\]

For a canonical calculus \( G \), \( \vdash_G \) is consistent iff \( G \) is coherent.
Characterization of constructiveness

**Constructive connective**

A connective is called constructive iff it can be defined by a coherent set of canonical rules.
A generalized Kripke-frame

A triple $W = \langle W, <, v \rangle$, where:

- $\langle W, < \rangle$ is a nonempty partially ordered set
- $v : W \times F \rightarrow \{t, f\}$ is a **persistent** function:
  
  if $v(w, \psi) = t$, then for every $w' \geq w$, $v(w', \psi) = t$.

- A sequent $\Gamma \Rightarrow \Delta$ is **locally true** in $w \in W$ if either
  
  $v(w, \psi) = f$ for some $\psi \in \Gamma$, or $v(w, \psi) = t$ for some $\psi \in \Delta$.

- A sequent is **true** in $w \in W$ if it is locally true in every $w' \geq w$.

- $W$ is a **model** of a sequent if it is locally true in every $w \in W$. 
G-legality of frames

Let $G$ be a canonical coherent single-conclusioned system. A generalized frame is G-legal if it respects the introduction and elimination rules of $G$.

Respecting introduction rules
The conclusion is locally true in $w \in W$ whenever the premises are true in $w$.

Respecting elimination rules
The conclusion is locally true in $w \in W$ whenever the definite premises are true in $w$ and the negative premises are locally true in $w$. 
Example 1

Implication rules:

\[
\{p_1 \Rightarrow p_2\} \ / \ \Rightarrow p_1 \supset p_2
\]

\[
v(w, \psi \supset \varphi) = t \text{ if } v(w', \psi) = f \text{ or } v(w', \varphi) = t \text{ for every } w' \geq w
\]

\[
\{\Rightarrow p_1 ; p_2 \Rightarrow\} \ / \ p_1 \supset p_2 \Rightarrow
\]

\[
v(w, \psi \supset \varphi) = f \text{ if } v(w, \psi) = t \text{ and } v(w, \varphi) = f
\]

The known semantics for intuitionistic implication!
Example 2

Semi-implication rules (Gurevich):

\[
\{ \Rightarrow p_2 \} / \Rightarrow p_1 \leadsto p_2
\]

\[\nu(w, \psi \leadsto \varphi) = t \text{ if } \nu(w, \varphi) = t\]

\[
\{ \Rightarrow p_1 ; p_2 \Rightarrow \} / p_1 \leadsto p_2 \Rightarrow
\]

\[\nu(w, \psi \leadsto \varphi) = f \text{ if } \nu(w, \psi) = t \text{ and } \nu(w, \varphi) = f\]

Non-deterministic (e.g., for the case when \(\nu(w', \psi) = \nu(w', \varphi) = f\) for every \(w' \geq w\))
Main results

**Soundness and completeness:**
A sequent $s$ is provable from a set of sequents $S$ in $G$ iff every $G$-legal frame which is a model of $S$ is also a model of $s$.

**Decidability:**
Every coherent canonical system is decidable.

**Cut-elimination:**
Every coherent canonical system admits strong cut-elimination.

**Modularity:**
The characterization of a constructive connective is independent of the system in which it is included.
Extension: basic systems

- Unlike in canonical systems, in basic sequent systems it is possible to control the context formulas.
- This allows one to have a larger variety of rules, and thus to handle more logics. For example,
  - Bi-intuitionistic logic:
    \[
    \frac{\Gamma, \psi \Rightarrow \varphi}{\Gamma \Rightarrow \psi \supset \varphi} \quad \frac{\Gamma \Rightarrow \psi, \Delta \quad \Gamma, \varphi \Rightarrow \Delta}{\Gamma, \psi \supset \varphi \Rightarrow \Delta}
    \]
    \[
    \frac{\psi \Rightarrow \varphi, \Delta}{\psi \prec \varphi \Rightarrow \Delta} \quad \frac{\Gamma \Rightarrow \psi, \Delta \quad \Gamma, \varphi \Rightarrow \Delta}{\Gamma \Rightarrow \psi \prec \varphi, \Delta}
    \]
  - The modal logic $K$:
    \[
    \frac{\Gamma \Rightarrow \psi}{\Box \Gamma \Rightarrow \Box \psi}
    \]
Basic systems - main results

- Every basic system has a (non-deterministic) Kripke-style semantics.
- In fact, there is a general method to obtain a (non-deterministic) Kripke-style semantics for a given basic systems.
- In addition, there are complete semantic characterizations of analyticity and (strong) cut-admissibility in basic systems.
Extension: canonical Gödel systems

- For some important logics, sequent systems do not suffice ⇒ hypersequent systems.
- A single-conclusion hypersequent is a set of single-conclusion sequents denoted by:
  \[ \Gamma_1 \Rightarrow E_1 \mid \Gamma_2 \Rightarrow E_2 \mid \ldots \mid \Gamma_n \Rightarrow E_n \]
- The only known “ideal” system for Gödel logic is the single-conclusion hypersequent system \( \text{HG} \) based on the rule:
  \[
  \frac{H \mid \Gamma, \Delta \Rightarrow E_1 \quad H \mid \Gamma, \Delta \Rightarrow E_2}{H \mid \Gamma \Rightarrow E_1 \mid \Delta \Rightarrow E_2} \quad (\text{com})
  \]
- Canonical Gödel systems: single-conclusion hypersequent systems with standard structural rules, (\text{com}), and any finite set of canonical single-conclusion logical rules.
Canonical Gödel systems - main results

- A general method to obtain (strongly) sound and complete Kripke semantics for canonical Gödel systems, based on linearly ordered frames.

- A general method to obtain (strongly) sound and complete many-valued semantics for canonical Gödel systems, based on the truth-values $[0, 1]$.

- The coherence criterion (from canonical single-conclusion sequent system) characterizes (strong) cut-admissibility in Canonical Gödel systems as well.
Non-deterministic semantics combined with Kripke-style frames are a powerful semantic formalism:

- Providing semantics for many natural classes of calculi (canonical single-conclusioned, basic, canonical Gödel, …)
- Semantic characterization of proof-theoretical properties of calculi.
Reminder: First-order languages

A first-order language $L$ includes:

- A set of variables $x_1, x_2, \ldots,$
- Parentheses, logical connectives (e.g. $\land$, $\lor$, $\supset$, $\neg$) and quantifiers (e.g., $\forall$ and $\exists$)
- The signature of $L$:
  - a (non-empty) set of predicate symbols
  - a set of constants
  - a set of function symbols
Matrices with unary quantifiers

\( M = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle \) is a (deterministic) matrix for a language \( L \) with unary quantifiers if:

1. \( \mathcal{V} \) is a nonempty set of truth-values,
2. \( \emptyset \neq \mathcal{D} \subset \mathcal{V} \) is a set of designated truth-values,
3. for every \( n \)-ary connective \( \Diamond \) of \( L \), \( \mathcal{O} \) includes an operation \( \tilde{\Diamond} : \mathcal{V}^n \rightarrow \mathcal{V} \),
4. for every unary quantifier \( Q \) of \( L \), \( \mathcal{O} \) includes an operation \( \tilde{Q} : P^+(\mathcal{V}) \rightarrow \mathcal{V} \).

Distribution quantifiers (coined by W.A. Carnielli)
## Example

<table>
<thead>
<tr>
<th>$H$</th>
<th>$\forall(H)$</th>
<th>$\exists(H)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>${t}$</td>
<td>$t$</td>
<td>$t$</td>
</tr>
<tr>
<td>${t,f}$</td>
<td>$f$</td>
<td>$t$</td>
</tr>
<tr>
<td>${f}$</td>
<td>$f$</td>
<td>$f$</td>
</tr>
</tbody>
</table>
Matrices: objectual quantification

- *Variables range over objects from the domain and assignments map variables to elements of the domain.*
Matrices: objectual quantification

- **Variables range over objects from the domain and assignments** map variables to elements of the domain.
- \( S = \langle D, I \rangle \) - an \( L \)-structure.
  - An assignment \( G \) in \( S \) maps the variables of \( L \) to \( D \).
  - Extend \( G \) to terms:

\[
G(c) = I(c), \quad G(f(t_1, \ldots, t_n)) = I(f)(G(t_1), \ldots, G(t_n))
\]
Matrices: objectual quantification

- **Variables range over objects from the domain and assignments map variables to elements of the domain.**
- $S = \langle D, I \rangle$ - an $L$-structure.
  - An *assignment* $G$ in $S$ maps the variables of $L$ to $D$.
  - Extend $G$ to terms:
    \[
    G(c) = I(c), \quad G(f(t_1, ..., t_n)) = I(f)(G(t_1), ..., G(t_n))
    \]

The valuation $\nu_{S,G}$

- $\nu_{S,G}(p(t_1, ..., t_n)) = I(p)(G(t_1), ..., G(t_n))$.
- $\nu_{S,G}(\diamond(\psi_1, ..., \psi_n)) = \tilde{\diamond}(\nu_{S,G}(\psi_1), ..., \nu_{S,G}(\psi_n))$.
- $\nu_{S,G}(Qx\psi) = \tilde{Q}({\nu_{S,G}\{x:=a\}(\psi) \mid a \in D})$.
  - where $G\{x:=a\}$ coincides with $G$ except for assigning $a \in D$ to $x$. 
Matrices: substitutional quantification

- **In classical first-order substitutional semantics,** a universally quantified sentence is true iff each of its substitution instances is true.

- **Assumption:** every element of the domain has a name.
  
  *Given an L-structure $S = \langle D, I \rangle$, extend the language with the set of individual constants $\{\overline{a} \mid a \in D\}$ interpreted as the corresponding domain elements.*
Matrices: substitutional quantification

- In classical first-order substitutional semantics, a universally quantified sentence is true iff each of its substitution instances is true.

- Assumption: every element of the domain has a name.

Given an $L$-structure $S = \langle D, I \rangle$, extend the language with the set of individual constants $\{a | a \in D\}$ interpreted as the corresponding domain elements.

The valuation $v_S$

- $v_S(p(t_1, \ldots, t_n)) = I(p)(I(t_1), \ldots, I(t_n))$
- $v_S(\diamond(\psi_1, \ldots, \psi_n)) = \tilde{\diamond}(v_S(\psi_1), \ldots, v_S(\psi_n))$
- $v_S(Qx\psi) = \tilde{Q}(\{v_S(\psi{\bar{a}/x}) | a \in D\})$
Nmatrices with unary quantifiers

\[ M = \langle V, D, O \rangle \] is a non-deterministic matrix (Nmatrix) for a language \( L \) with unary quantifiers if:

1. \( V \) is a nonempty set of truth-values,
2. \( \emptyset \neq D \subseteq V \) is a set of designated truth-values,
3. for every \( n \)-ary connective \( \Diamond \) of \( L \), \( O \) includes an operation \( \tilde{\Diamond} : V^n \to P^+(V) \),
4. for every unary quantifier \( Q \) of \( L \), \( O \) includes an operation \( \tilde{Q} : P^+(V) \to P^+(V) \).
Example

Consider the two-valued Nmatrix $\mathcal{M}_1 = \langle \{ t, f \}, \{ t \}, \mathcal{O} \rangle$ for a language $L$ over $\{ Q, \forall, \neg \}$, where $\mathcal{O}$ contains the following operations:

<table>
<thead>
<tr>
<th>$H$</th>
<th>$\tilde{Q}(H)$</th>
<th>$H$</th>
<th>$\tilde{\forall}(H)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>${ t }$</td>
<td>${ t }$</td>
<td>${ t }$</td>
<td>${ t }$</td>
</tr>
<tr>
<td>${ t, f }$</td>
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<td>${ f }$</td>
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<tr>
<td>${ f }$</td>
<td>${ f }$</td>
<td>${ f }$</td>
<td>${ f }$</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>$a$</th>
<th>$\neg a$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t$</td>
<td>${ t, f }$</td>
</tr>
<tr>
<td>$f$</td>
<td>${ t }$</td>
</tr>
</tbody>
</table>
Nmatrices: objectual quantification

\[ \nu_{S,G}(p(t_1, \ldots, t_n)) = I(p)(G(t_1), \ldots, G(t_n)). \]
Nmatrices: objectual quantification

\[ \nu_{S,G}(p(t_1, \ldots, t_n)) = I(p)(G(t_1), \ldots, G(t_n)). \]

\[ \nu_{S,G}(\diamond(\psi_1, \ldots, \psi_n)) \in \tilde{\diamond}(\nu_{S,G})(\psi_1), \ldots, \nu_{S,G}(\psi_n)). \]
Nmatrices: objectual quantification

- $\nu_{S,G}(p(t_1, ..., t_n)) = I(p)(G(t_1), ..., G(t_n))$.
- $\nu_{S,G}(<\psi_1, ..., \psi_n>) \in \tilde{\Diamond}(\nu_{S,G}(\psi_1), ..., \nu_{S,G}(\psi_n))$.
- $\nu_{S,G}(Q \times \psi) \in \tilde{Q}[\{\nu_{S,G}[x:=a](\psi) \mid a \in D\}]$.  

???
Substitutional quantification

Reminder: For $S = \langle D, I \rangle$, the language extended by individual constants is denoted by $L(D)$

Let $S = \langle D, I \rangle$ be an $L$-structure. A valuation in an Nmatrix $M$ for $L$ is a function $v$ from sentences of $L(D)$ to $\mathcal{V}$, satisfying:

- $v((p(t_1, \ldots, t_n)) = I(p)(I(t_1), \ldots, I(t_n))$
- $v(\forall (\psi_1, \ldots, \psi_n)) \in \tilde{\forall}(v(\psi_1), \ldots, v(\psi_n))$
- $v(Qx\psi) \in \tilde{Q}({v(\psi\{\overline{a}/x\}) \mid a \in D})$
The problem of $\alpha$-equivalence

- $\psi \equiv_\alpha \psi'$ if $\psi$ can be obtained from $\psi'$ by renaming bound variables.
- Problem: two $\alpha$-equivalent sentences are not necessarily assigned the same truth-value.
- Example:

<table>
<thead>
<tr>
<th>$H$</th>
<th>$\neg\forall[H]$</th>
</tr>
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<tbody>
<tr>
<td>{t}</td>
<td>{t}</td>
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<td>{t,f}</td>
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<td>f</td>
<td>{t}</td>
</tr>
</tbody>
</table>

Consider: $\neg\forall xp(x)$ and $\neg\forall yp(y)$
Definition of a non-deterministic valuation - corrected

Let $S = \langle D, I \rangle$ be an $L$-structure. A valuation in an Nmatrix $\mathcal{M}$ for $L$ is a function $v$ from closed sentences of $L(D)$ to $\mathcal{V}$ satisfying:

- $v(p(t_1, \ldots, t_n)) = I(p)(I(t_1), \ldots, I(t_n))$.
- $v(\top(\psi_1, \ldots, \psi_n)) \in \tilde{\top}(v(\psi_1), \ldots, v(\psi_n))$.
- $v(Qx\psi) \in \tilde{Q}(\{v(\psi{\bar{a}/x}) \mid a \in D\})$.
- If $\psi_1 \equiv_\alpha \psi_2$, then $v(\psi_1) = v(\psi_2)$.
Other problems to handle

- Terms denoting the same objects cannot be used interchangeably.
- Void quantification for first-order quantifiers $\forall$ and $\exists$.
- Example:

<table>
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</tr>
<tr>
<td>f</td>
<td>{t}</td>
</tr>
</tbody>
</table>

Let $S = \langle \{1, 2\}, I \rangle$, $I(p)(1) = I(p)(2) = t$ and $I(c) = I(d) = 1$.

Consider: (i) $\neg p(c)$ and $\neg p(d)$, (ii) $\neg \forall x p(c)$ and $\neg p(c)$.

Solution: add appropriate congruence relations. For instance, $A \sim_{void} QxA$ if $x \not\in Fv(A)$. 
Analyticity

Analyticity of an Nmatrix $M$

for every $L$-structure $S$ and every partial $M$-legal $S$-valuation $v_p$
defined on a set of $L$-sentences closed under subformulas: $v_p$ can
be extended to a full $M$-legal valuation.
Analyticity

Analyticity of an Nmatrix $M$

for every $L$-structure $S$ and every partial $M$-legal $S$-valuation $v_p$ defined on a set of $L$-sentences closed under subformulas: $v_p$ can be extended to a full $M$-legal valuation.

- Analyticity is not guaranteed anymore when congruence relations are involved.
- Some good cases:
  - Analyticity for $\equiv_\alpha$ is always guaranteed.
  - Denote $\varphi_1 \sim^{dc} \varphi_2$ if $\varphi_2$ can be obtained from $\varphi_1$ by renaming bound variables and deleting/adding void quantifiers. Analyticity for $\sim^{dc}$ is guaranteed iff $a \in \tilde{Q}_M(\{a\})$ for every quantifier $Q$ of $L$ and every $a \in \mathcal{V}$. 
Using congruences in the propositional case

- Introducing congruences can be useful also in the propositional case (e.g. equivalence in all contexts of $\psi \land \varphi$ and $\varphi \land \psi$).
- Analyticity should be handled with care (question for further research)
Application: first-order C-systems

Language: $L_{QC} = \{\land, \lor, \supset, \neg, \lozenge, \forall, \exists\}$.

Logic: QBK is obtained by adding the following axioms to some standard Hilbert-type system for classical positive FOL:

(N1) $\neg \varphi \lor \varphi$

(b) $(\lozenge \varphi \land \varphi \land \neg \varphi) \supset \psi$

(k) $\lozenge \psi \lor (\psi \land \neg \psi)$

(DC) $\varphi_1 \supset \varphi_2$ whenever $\varphi_1 \sim^{dc} \varphi_2$.

$\varphi_1 \sim^{dc} \varphi_2$ if $\varphi_2$ can be obtained from $\varphi_1$ by renaming bound variables and deleting/adding void quantifiers.
Extensions of QBK

(c) \( \neg \neg \varphi \subseteq \varphi \)

(e) \( \varphi \subseteq \neg \neg \varphi \)

\[
\ldots
\]

(a\(\forall\)) \( \forall x \circ \varphi \subseteq o(\forall x \varphi) \)

(a\(\exists\)) \( \forall x \circ \varphi \subseteq o(\exists x \varphi) \)

(o\(\forall\)) \( \exists x \circ \varphi \subseteq o(\forall x \varphi) \)

(o\(\exists\)) \( \exists x \circ \varphi \subseteq o(\exists x \varphi) \)

Example: da-Costa’s original \( C_1^* \) is equivalent to QBKcilə.
The idea of semantics

- Truth-value: $v(\varphi) = \langle x, y \rangle$, where $x$ expresses truth/falsity of $\varphi$ and $y$ expresses truth/falsity of $\neg \varphi$. 
The idea of semantics

- **Truth-value:** \( v(\varphi) = \langle x, y \rangle \), where \( x \) expresses truth/falsity of \( \varphi \) and \( y \) expresses truth/falsity of \( \neg \varphi \).

- **Possible values:**
  - \( v(\varphi) = \langle 1, 0 \rangle = t \) - \( \varphi \) is true and \( \neg \varphi \) is false
  - \( v(\varphi) = \langle 0, 1 \rangle = f \) - \( \varphi \) is false and \( \neg \varphi \) is true
  - \( v(\varphi) = \langle 1, 1 \rangle = \top \) - \( \varphi \) is true and \( \neg \varphi \) is true

- **Addition:** Every \( M \)-legal valuation should also respect the congruences for \( \alpha \)-equivalence and void quantification (but analyticity is preserved!).
3-valued Semantics for QBK

The Nmatrix $QM = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ is defined by:

$\mathcal{V} = \{ t, \top, f \}$, $\mathcal{D} = \{ t, \top \}$, and $\mathcal{F} = \{ f \}$:

$a \tilde{\land} b = \begin{cases} D & \text{if } a \in D \text{ and } b \in D \\ F & \text{if } a \in F \text{ or } b \in F \end{cases}$

$a \tilde{\lor} b = \begin{cases} D & \text{if } a \in D \text{ or } b \in D \\ F & \text{if } a \in F \text{ and } b \in F \end{cases}$

$\tilde{\neg} a = \begin{cases} D & \text{if } a \in \{ \top, f \} \\ F & \text{if } a = t \end{cases}$

$\tilde{\forall}(H) = \begin{cases} D & \text{if } H \subseteq D \\ F & \text{otherwise} \end{cases}$

$\tilde{\exists}(H) = \begin{cases} D & \text{if } H \cap D \neq \emptyset \\ F & \text{otherwise} \end{cases}$
Effects of \((a_Q)\) and \((o_Q)\) for \(Q \in \{\forall, \exists\}\)

\[
\begin{align*}
(a_\forall) \quad & \forall x \circ \varphi \supset o(\forall x \varphi) \\
(a_\exists) \quad & \forall x \circ \varphi \supset o(\exists x \varphi) \\
o_\forall \quad & \exists x \circ \varphi \supset o(\forall x \varphi) \\
o_\exists \quad & \exists x \circ \varphi \supset o(\exists x \varphi)
\end{align*}
\]

\[
\begin{align*}
\text{Cond}(a_\forall) : & \forall^{\circ}(\{t\}) = \{t\} \\
\text{Cond}(a_\exists) : & \exists^{\circ}(\{t\}) = \exists^{\circ}(\{t, f\}) = \{t\} \\
\text{Cond}(o_\forall) : & \forall^{\circ}(\{t\}) = \exists^{\circ}(\{t, T\}) = \{t\} \\
\text{Cond}(o_\exists) : & \exists^{\circ}(\{t\}) = \exists^{\circ}(\{t, T\}) = \\
& \exists^{\circ}(\{t, f\}) = \exists^{\circ}(\{t, T, f\}) = \{t\}
\end{align*}
\]
The Nmatrix $\mathbf{M}_{C_1^*}$

$\mathcal{V} = \mathcal{T} \cup \mathcal{I} \cup \mathcal{F}, \quad \mathcal{T} = \{ t^i_j \mid i \geq 0, j \geq 0 \}, \quad \mathcal{I} = \{ \top^i_j \mid i \geq 0, j \geq 0 \}, \quad \mathcal{F} = \{ f \}, \quad \mathcal{D} = \mathcal{T} \cup \mathcal{I}.$

$a \tilde{\lor} b = \begin{cases} \mathcal{F} & \text{if } a \in \mathcal{D} \text{ and } b \in \mathcal{F} \\ \mathcal{T} & \text{if } a \in \mathcal{F} \text{ and } b \notin \mathcal{I}, \text{ or} \\ \mathcal{D} & \text{if } b \in \mathcal{T} \text{ and } a \notin \mathcal{I} \\ \text{otherwise} \end{cases}$

$a \tilde{\land} b = \begin{cases} \mathcal{F} & \text{if } a \in \mathcal{F} \text{ or } b \in \mathcal{F} \\ \mathcal{T} & \text{if } a \in \mathcal{T} \text{ and } b \in \mathcal{T}, \text{ or} \\ \mathcal{D} & \text{if } a = \top^i_j \text{ and } b \in \{ \top^i_j+1, t^i_{j+1} \} \\ \text{otherwise} \end{cases}$

$\tilde{\neg} a = \begin{cases} \mathcal{F} & \text{if } a \in \mathcal{T} \\ \mathcal{T} & \text{if } a \in \mathcal{F} \\ \{ \top^i_j+1, t^i_{j+1} \} & \text{if } a = \top^i_j \end{cases}$

$\tilde{\forall}(H) = \begin{cases} \mathcal{T} & \text{if } H \subseteq \mathcal{T} \\ \mathcal{D} & \text{if } H \subseteq \mathcal{D} \text{ and } H \cap \mathcal{I} \neq \emptyset \\ \mathcal{F} & f \in H \end{cases}$

$\tilde{\exists}(H) = \begin{cases} \mathcal{T} & \text{if } H \subseteq \mathcal{T} \cup \mathcal{F} \text{ and } H \cap \mathcal{T} \neq \emptyset \\ \mathcal{D} & \text{if } H \cap \mathcal{I} \neq \emptyset \\ \mathcal{F} & H = \{ f \} \end{cases}$
Application: \( \neg \exists x \neg p(x) \not\vdash_{C_1^*} \forall x p(x) \)

- A much easier semantic proof: refutation using \( M_{C_1^*} \).

\[ S = \langle \{a, b\}, I \rangle \]

\[ I(p)(a) = \top_0 \quad I(p)(b) = f \]

Next define a partial valuation \( \nu \) on the set of subformulas of \( \{\neg \exists x \neg p(x), \forall x p(x)\} \) as follows:

\[ \nu(p(a)) = \top_0 \quad \nu(p(b)) = f \quad \nu(\neg p(a)) = \top_0 \quad \nu(\neg p(b)) = t_0 \]

\[ \nu(\exists x \neg p(x)) = \top_0 \quad \nu(\neg \exists x \neg p(x)) = t_0^2 \quad \nu(\forall x p(x)) = f \]

\( \nu \) is \( M_{C_1^*} \)-legal, and (by the analyticity of \( M_{C_1^*} \)) it can be extended to a full \( M_{C_1^*} \)-legal valuation.
Reminder: propositional canonical systems

Each logical rule satisfies:

1. Introduces exactly one formula in its conclusion.
2. The introduced formula: $\diamond(\psi_1, \ldots, \psi_n)$.
3. All active formulas in its premises are in $\{\psi_1, \ldots, \psi_n\}$.
4. No restrictions on the side formulas.

Direct correspondence: A canonical system is coherent iff it admits cut-elimination iff it has a characteristic $2N$-matrix.
Canonical quantifier rules

\[ \Gamma, A\{t/w\} \Rightarrow \Delta \quad \frac{\Gamma \Rightarrow A\{z/w\}, \Delta}{\Gamma \Rightarrow \forall w A, \Delta} \]

where \( z \) is a variable free for \( w \) in \( A \), \( z \) is not free in \( \Gamma \cup \Delta \cup \{\forall w A\} \), and \( t \) is any term free for \( w \) in \( A \).

\[ \Downarrow \]

\[ \frac{A\{t/w\} \Rightarrow}{\forall w A \Rightarrow} \quad \frac{\Rightarrow A\{z/w\}}{\Rightarrow \forall w A} \]

\[ \Downarrow \]

\[ \{p(c) \Rightarrow\}/\forall w p(w) \Rightarrow \quad \{\Rightarrow p(y)\}/ \Rightarrow \forall w p(w) \]

An eigenvariable is marked by a variable, and a term is marked by a constant.
Constructive Canonical Systems
FOL Defs
FO C-systems
Canonical Systems with Quantifiers

Canonical systems

A canonical system includes

1. Axioms: $\psi \Rightarrow \psi'$ for $\psi \equiv_\alpha \psi'$
2. Structural Weakening and Cut rules:
   \[
   \Gamma \Rightarrow \Delta \quad (\text{Weakening})
   \]
   \[
   \Gamma, \psi \Rightarrow \Delta, \psi \Rightarrow \Delta \quad (\text{Cut})
   \]
3. Substitution rule:
   \[
   \Gamma \Rightarrow \Delta \quad (S)
   \]
   
   where $\Gamma', \Delta'$ are substitution instances of $\Gamma, \Delta$ resp.
Coherence

- A canonical calculus $G$ is **coherent** if for every two canonical rules of $G$ of the form $\Theta_1/\Rightarrow A$ and $\Theta_2/ A \Rightarrow$, the set of clauses $\Theta_1 \cup \Theta_2$ is classically inconsistent.
- *The coherence of a canonical calculus $G$ is decidable.*
- **Examples:**
  - Coherent:
    - $\{ p(c) \Rightarrow \} / \forall x \ p(x) \Rightarrow \ \{ \Rightarrow p(y) \} / \Rightarrow \forall x \ p(x)$
  - *Non-coherent:*
    - $\{ \Rightarrow p(c) \} / \Rightarrow Qxp(x) \quad \{ p(d) \Rightarrow \} / \Rightarrow Qxp(x)$
Correspondence Theorem

The following statements concerning a canonical system $G$ with unary quantifiers are equivalent:

1. $G$ is coherent.
2. $G$ has a characteristic 2Nmatrix.

**Strong cut-elimination**

$G$ admits **strong cut-elimination** if whenever $S \vdash s$, then $s$ has a proof from $S$ in $G$, where cuts are applied only on substitution instances of formulas from $S$. 

More General Quantifiers

- A natural step: \( n \)-ary quantifiers:
  
  *If \( Q \) is an \( n \)-ary quantifier, then \( Qx(\psi_1, \ldots, \psi_n) \) is a formula.*

- Examples:
  
  1. **Unary quantifiers**: \( \forall, \exists \).
  2. **Binary quantifiers**: bounded universal and existential quantifiers \( \forall \) and \( \exists \), where:
     
     - \( \forall(\psi_1, \psi_2) \) means \( \forall x(\psi_1 \rightarrow \psi_2) \).
     - \( \exists(\psi_1, \psi_2) \) means \( \exists x(\psi_1 \land \psi_2) \).
Nmatrices with $n$-ary quantifiers

- An $n$-ary quantifier $Q$ in an Nmatrix $M = \langle V, D, O \rangle$ is interpreted by a function $\tilde{Q}: P^+(V^n) \to P^+(V)$.
- Example: for every $E \in P^+(\{t, f\}^2)$:

$$\tilde{\forall}(E) = \begin{cases} \{f\} & \text{if } \langle t, f \rangle \in E \\ \{t\} & \text{otherwise} \end{cases} \quad \tilde{\exists}(E) = \begin{cases} \{t\} & \text{if } \langle t, t \rangle \in E \\ \{f\} & \text{otherwise} \end{cases}$$

The definition of an $M$-valuation $\nu$ is now modified as follows:

$$\nu(Qx(\psi_1, ..., \psi_n)) \in \tilde{Q}_M(\{\langle v(\psi_1\{a/x\}), ..., v(\psi_n\{a/x\}) \rangle \mid a \in D\})$$

The framework of canonical systems can be extended to the case of $n$-ary quantifiers, the direct correspondence still holds.
### Example

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<th>$\mathbf{H}$</th>
<th>$\neg (\mathbf{H})$</th>
<th>$\exists [\mathbf{H}]$</th>
<th>$\tilde{Q}_2 [\mathbf{H}]$</th>
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Summary

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  - Characterization of various non-classical logics
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  - Characterization of various non-classical logics
- Allows for a **systematic and modular** approach
- Insights into the syntax-semantics interface
- Provides important tools for Universal Logic.

Thank you for your attention!