# CANONICAL CALCULI WITH (N,K)-ARY QUANTIFIERS

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ABSTRACT. Propositional canonical Gentzen-type systems, introduced in [2], are systems which in addition to the standard axioms and structural rules have only logical rules in which exactly one occurrence of a connective is introduced and no other connective is mentioned. [2] provides a constructive coherence criterion for the non-triviality of such systems and shows that a system of this kind admits cut-elimination iff it is coherent. The semantics of such systems is provided using two-valued non-deterministic matrices (2Nmatrices). [23] extends these results to systems with unary quantifiers of a very restricted form. In this paper we substantially extend the characterization of canonical systems to $(n,k)$-ary quantifiers, which bind $k$ distinct variables and connect $n$ formulas, and show that the coherence criterion remains constructive for such systems. Then we focus on the case of $k \in \{0,1\}$ and show that the following statements concerning a canonical calculus $G$ are equivalent: (i) $G$ is coherent, (ii) $G$ has a strongly characteristic 2Nmatrix, and (iii) $G$ admits strong cut-elimination. We also show that coherence is not a necessary condition for standard cut-elimination, and then characterize a subclass of canonical systems for which this property does hold.

## INTRODUCTION

An $(n,k)$-*ary quantifier* (for $n > 0$, $k \geq 0$) is a generalized logical connective, which binds $k$ variables and connects $n$ formulas. Any $n$-ary propositional connective can be thought of as an $(n,0)$-ary quantifier. For instance, the standard $\wedge$ connective binds no variables and connects two formulas: $\wedge(\psi_1,\psi_2)$. The standard first-order quantifiers $\exists$ and $\forall$ are $(1,1)$-quantifiers, as they bind one variable and connect one formula: $\forall x\psi, \exists x\psi$. Bounded universal and existential quantifiers used in syllogistic reasoning ($\forall x(p(x) \rightarrow q(x))$ and $\exists x(p(x) \wedge q(x))$) can be represented as (2,1)-ary quantifiers $\overline{\forall}$ and $\overline{\exists}$, binding one variable and connecting two formulas: $\overline{\forall}x(p(x), q(x))$ and $\overline{\exists}x(p(x), q(x))$. An example of $(n,k)$-ary quantifiers for $k > 1$ are Henkin quantifiers[1] ([14, 15]). The simplest Henkin quantifier $Q_H$

*2000 ACM Subject Classification:* F.4.1.

*Key words and phrases:* Proof Theory, Automated Deduction, Cut Elimination, Gentzen-type Systems, Quantifiers, Many-valued Logic, Non-deterministic Matrices.

[1]It should be noted that the semantic interpretation of quantifiers used in this paper is not sufficient for treating such quantifiers.

LOGICAL METHODS
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DOI:10.2168/LMCS-???

binds 4 variables and connects one formula:

$$Q_H\ x_1x_2y_1y_2\ \psi(x_1,x_2,y_1,y_2) := \begin{matrix} \forall x_1 & \exists y_1 \\ \forall x_2 & \exists y_2 \end{matrix}\ \psi(x_1,x_2,y_1,y_2)$$

In this way of recording combinations of quantifiers, dependency relations between variables are expressed as follows: an existentially quantified variable depends on those universally quantified variables which are on the left of it in the same row.

According to a long tradition in the philosophy of logic, established by Gentzen in his classical paper *Investigations Into Logical Deduction* ([12]), an "ideal" set of introduction rules for a logical connective should determine the *meaning* of the connective. In [2, 3] the notion of a "canonical propositional Gentzen-type rule" was first defined in precise terms. A constructive *coherence* criterion for the non-triviality of systems consisting of such rules was provided, and it was shown that a system of this kind admits cut-elimination iff it is coherent. It was further proved that the semantics of such systems is provided by two-valued non-deterministic matrices (2Nmatrices), which form a natural generalization of the classical matrix. In fact, a characteristic 2Nmatrix was constructed for every coherent canonical propositional system.

In [23] the results were extended to systems (of a restricted form) with unary quantifiers. A characterization of a "canonical unary quantificational rule" in such calculi was proposed (the standard Gentzen-type rules for $\forall$ and $\exists$ are canonical according to it), and a constructive extension of the coherence criterion of [2, 3] for canonical systems of this type was given. 2Nmatrices were extended to languages with unary quantifiers, using a *distributional* interpretation of quantifiers ([18],[6]). Then it was proved that again a canonical Gentzen-type system of this type admits cut-elimination iff it is coherent, and that it is coherent iff it has a characteristic 2Nmatrix.

In this paper we make the intuitive notion of a "well-behaved" introduction rule for $(n, k)$-ary quantifiers formally precise. We considerably extend the scope of the characterizations of [2, 3, 23] to "canonical $(n, k)$-ary quantificational rules", so that both the propositional systems of [2, 3] and the restricted quantificational systems of [23] are specific instances of the proposed definition. We show that the coherence criterion for the defined systems remains decidable. Then we focus on the case of $k \in \{0, 1\}$ and show that the following statements concerning a canonical calculus $G$ are equivalent: (i) $G$ is coherent, (ii) $G$ has a strongly characteristic 2Nmatrix, and (iii) $G$ admits strong cut-elimination. We show that coherence is not a necessary condition for standard cut-elimination, and then characterize a subclass of canonical systems for which this property does hold.

## 1. Preliminaries

For any $n > 0$ and $k \geq 0$, if a quantifier $\mathcal{Q}$ is of arity $(n, k)$, then $\mathcal{Q}x_1...x_k(\psi_1, ..., \psi_n)$ is a formula whenever $x_1, ..., x_k$ are distinct variables and $\psi_1, ..., \psi_n$ are formulas of $L$.
For interpretation of quantifiers, we use a generalized notion of *distributions* (see, e.g [18, 6]). Given a set $S$, $P^+(S)$ is the set of all the nonempty subsets of $S$.

**Definition 1.1.** Given a set of truth value $\mathcal{V}$, a *distribution* of a (1,1)-ary quantifier $\mathcal{Q}$ is a function $\lambda_{\mathcal{Q}} : P^+(\mathcal{V}) \to \mathcal{V}$.

(1,1)-ary distribution quantifiers have been extensively studied and axiomatized in many-valued logic. See, for instance, [6, 19, 13].

In what follows, $L$ is a language with $(n, k)$-ary quantifiers, that is with quantifiers

$\mathcal{Q}_1, ..., \mathcal{Q}_m$ with arities $(n_1, k_1)$, ..., $(n_m, k_m)$ respectively. Denote by $Frm_L^{cl}$ the set of closed $L$-formulas and by $Trm_L^{cl}$ the set of closed $L$-terms. $Var = \{v_1, v_2, ...,\}$ is the set of variables of $L$. We use the metavariables $x, y, z$ to range over elements of $Var$.

$\equiv_\alpha$ is the $\alpha$-equivalence relation between formulas, i.e identity up to the renaming of bound variables.

**Lemma 1.2.** *Let $\mathcal{Q}$ be an $(n, k)$-ary quantifier of $L$ and $z_1, ..., z_k$ fresh variables which do not occur in $\mathcal{Q}x_1..x_k(\psi_1, ..., \psi_n)$. Then: $\mathcal{Q}x_1...x_k(\psi_1, ..., \psi_n) \equiv_\alpha \mathcal{Q}y_1...y_k(\psi'_1, ..., \psi'_n)$ iff $\psi_i\{z_1/x_1, ..., z_k/x_k\} \equiv_\alpha \psi'_i\{z_1/y_1, ..., z_k/y_k\}$ for every $1 \leq i \leq n$.*

The proof is not hard and is left to the reader.

We use [ ] for application of functions in the meta-language, leaving the use of ( ) to the object language. $A\{\mathbf{t}/x\}$ denotes the formula obtained from $A$ by substituting $\mathbf{t}$ for $x$. Given an $L$-formula $A$, $Fv[A]$ is the set of variables occurring free in $A$. We denote $\mathcal{Q}x_1...x_kA$ by $\mathcal{Q}\overrightarrow{x}A$, and $A(x_1, ..., x_k)$ by $A(\overrightarrow{x})$.
A set of sequents $\mathcal{S}$ satisfies the *free-variable condition* if the set of variables occurring bound in $\mathcal{S}$ is disjoint from the set of variables occurring free in $\mathcal{S}$.

## 2. Canonical Systems with (n,k)-ary quantifiers

In this section we propose a precise characterization of a "canonical $(n, k)$-ary quantificational Gentzen-type rule".
Using an introduction rule for an $(n, k)$-ary quantifier $\mathcal{Q}$, we should be able to derive a sequent of the form $\Gamma \Rightarrow \mathcal{Q}x_1...x_k(\psi_1, ..., \psi_n), \Delta$ or of the form $\Gamma, Qx_1...x_k(\psi_1, ..., \psi_n) \Rightarrow \Delta$, based on some information about the subformulas of $\mathcal{Q}x_1...x_k(\psi_1, ..., \psi_n)$ contained in the premises of the rule. For instance, consider the following standard rules for the (1,1)-ary quantifier $\forall$:

$$\frac{\Gamma, A\{\mathbf{t}/w\} \Rightarrow \Delta}{\Gamma, \forall w\, A \Rightarrow \Delta}\ (\forall \Rightarrow) \qquad \frac{\Gamma \Rightarrow A\{z/w\}, \Delta}{\Gamma \Rightarrow \forall w\, A, \Delta}\ (\Rightarrow \forall)$$

where $\mathbf{t}, z$ are free for $w$ in $A$ and $z$ does not occur free in the conclusion. Our key observation is that the internal structure of $A$, as well as the exact term $\mathbf{t}$ or variable $w$ used, are immaterial for the meaning of $\forall$. What is important here is the sequent on which $A$ appears, as well as whether a term variable $\mathbf{t}$ or an eigenvariable $z$ is used.
It follows that the internal structure of the formulas of $L$ used in the description of a rule can be abstracted by using a simplified first-order language, i.e. the formulas of $L$ in an introduction rule of a $(n, k)$-ary quantifier, can be represented by *atomic* formulas with predicate symbols of arity $k$. The case when the substituted term is any $L$-term, will be signified by a constant, and the case when it is a variable satisfying the above conditions - by a variable. In other words, constants serve as term variables, while variables are eigenvariables.
Thus in addition to our original language $L$ with $(n, k)$-ary quantifiers we define another, simplified language.

**Definition 2.1.** For $k \geq 0$, $n \geq 1$ and a set of constants $Con$, $L_k^n(Con)$ is the (first-order) language with $n$ $k$-ary predicate symbols $p_1, ..., p_n$ and the set of constants $Con$ (and no quantifiers). The set of variables of $L_k^n(Con)$ is $Var = \{v_1, v_2, ...,\}$.

Note that $L^n_k(Con)$ and $L$ share the same set of variables. Furthermore, henceforth we assume that for every $(n,k)$-ary quantifier $\mathcal{Q}$ of $L$, $L^n_k(Con)$ is a subset of $L$. This assumption is not necessary, but it makes the presentation easier, as will be explained in the sequel.

Next we formalize the notion of a canonical rule and its application.

**Definition 2.2.** Let $Con$ be some set of constants. A *canonical quantificational rule of arity* $(n,k)$ is an expression of the form $\{\Pi_i \Rightarrow \Sigma_i\}_{1\le i\le m}/C$, where $m \ge 0$, $C$ is either $\Rightarrow \mathcal{Q}v_1...v_k(p_1(v_1,...,v_k),...,p_n(v_1,...,v_k))$ or $\mathcal{Q}v_1...v_k(p_1(v_1,...,v_k),...,p_n(v_1,...,v_k)) \Rightarrow$ for some $(n,k)$-ary quantifier $\mathcal{Q}$ of $L$ and for every $1 \le i \le m$: $\Pi_i \Rightarrow \Sigma_i$ is a clause[2] over $L^n_k(Con)$.

Henceforth, in cases where the set of constants $Con$ is clear from the context (it is the set of all constants occurring in a canonical rule), we will write $L^n_k$ instead of $L^n_k(Con)$.

A canonical rule is a schematic representation, while for an actual application we need to instantiate the schematic variables by the terms and formulas of $L$. This is done using a mapping function, defined as follows.

**Definition 2.3.** Let $R = \Theta/C$ be an $(n,k)$-ary canonical rule, where $C$ is of one of the forms $(\mathcal{Q}\overrightarrow{v}(p_1(\overrightarrow{v}),...,p_n(\overrightarrow{v})) \Rightarrow)$ or $(\Rightarrow \mathcal{Q}\overrightarrow{v}(p_1(\overrightarrow{v}),...,p_n(\overrightarrow{v})))$. Let $\Gamma$ be a set of $L$-formulas and $z_1,...,z_k$ - distinct variables of $L$. An $\langle R,\Gamma,z_1,...,z_k\rangle$*-mapping* is any function $\chi$ from the predicate symbols, terms and formulas of $L^n_k$ to formulas and terms of $L$, satisfying the following conditions:

- For every $1 \le i \le n$, $\chi[p_i]$ is an $L$-formula.
- $\chi[y]$ is a variable of $L$.
- $\chi[x] \ne \chi[y]$ for every two variables $x \ne y$.
- $\chi[c]$ is an $L$-term, such that $\chi[x]$ does not occur in $\chi[c]$ for any variable $x$ occurring in $\Theta$.
- For every $1 \le i \le n$, whenever $p_i(\mathbf{t}_1,...,\mathbf{t}_k)$ occurs in $\Theta$, for every $1 \le j \le k$: $\chi[\mathbf{t}_j]$ is a term free for $z_j$ in $\chi[p_i]$, and if $\mathbf{t}_j$ is a variable, then $\chi[\mathbf{t}_j]$ does not occur free in $\Gamma \cup \{\mathcal{Q}z_1...z_k(\chi[p_1],...,\chi[p_n])\}$.
- $\chi[p_i(\mathbf{t}_1,...,\mathbf{t}_k)] = \chi[p_i]\{\chi[\mathbf{t}_1]/z_1,...,\chi[\mathbf{t}_k]/z_k\}$.

We extend $\chi$ to sets of $L^n_k(Con_\Theta)$-formulas as follows:

$$\chi[\Delta] = \{\chi[\psi] \mid \psi \in \Delta\}$$

Given a schematic representation of a rule and an instantiation mapping, we can define an application of a rule as follows.

**Definition 2.4.** An *application* of a canonical rule of arity $(n,k)$
$R = \{\Pi_i \Rightarrow \Sigma_i\}_{1\le i\le m}/\mathcal{Q}\overrightarrow{v}(p_1(\overrightarrow{v}),...,p_n(\overrightarrow{v})) \Rightarrow$ is any inference step of the form:

$$\frac{\{\Gamma,\chi[\Pi_i] \Rightarrow \Delta,\chi[\Sigma_i]\}_{1\le i\le m}}{\Gamma,\mathcal{Q}z_1...z_k\ (\chi[p_1],...,\chi[p_n]) \Rightarrow \Delta}$$

where $z_1,...,z_k$ are variables, $\Gamma,\Delta$ are any sets of $L$-formulas and $\chi$ is some $\langle R,\Gamma\cup\Delta,z_1,...,z_k\rangle$-mapping.

[2]By a clause we mean a sequent containing only atomic formulas.

An application of a canonical quantificational rule of the form $\{\Pi_i \Rightarrow \Sigma_i\}_{1\leq i\leq m}/ \Rightarrow \mathcal{Q}\overrightarrow{v}(p_1(\overrightarrow{v}),...,p_n(\overrightarrow{v}))$ is defined similarly.

Below we demonstrate the above definition by a number of examples.

**Examples 2.5.** (1) The standard right introduction rule for $\wedge$, which can be thought of as an $(2,0)$-ary quantifier is $\{\Rightarrow p_1, \Rightarrow p_2\}/ \Rightarrow p_1 \wedge p_2$. Its application is of the form:

$$\frac{\Gamma \Rightarrow \psi_1, \Delta \quad \Gamma \Rightarrow \psi_2, \Delta}{\Gamma \Rightarrow \psi_1 \wedge \psi_2, \Delta}$$

(2) The standard introduction rules for the $(1,1)$-ary quantifiers $\forall$ and $\exists$ can be formulated as follows:

$$\{p_1(c) \Rightarrow\}/\forall v_1\, p_1(v_1) \Rightarrow \qquad \{\Rightarrow p_1(v_1)\}/ \Rightarrow \forall v_1\, p_1(v_1)$$

$$\{\Rightarrow p_1(d)\}/ \Rightarrow \exists v_1\, p_1(v_1) \qquad \{p_1(v_1) \Rightarrow\}/\exists v_1\, p_1(v_1) \Rightarrow$$

Applications of these rules have the forms:

$$\frac{\Gamma, \psi\{\mathbf{t}/w\} \Rightarrow \Delta}{\Gamma, \forall w\, \psi \Rightarrow \Delta}\ (\forall \Rightarrow) \qquad \frac{\Gamma \Rightarrow \psi\{z/w\}, \Delta}{\Gamma \Rightarrow \forall w\, \psi, \Delta}\ (\Rightarrow \forall)$$

$$\frac{\Gamma \Rightarrow \psi\{\mathbf{t}/w\}, \Delta}{\Gamma \Rightarrow \exists w\, A, \Delta}\ (\Rightarrow \exists) \qquad \frac{\Gamma, \psi\{z/w\} \Rightarrow \Delta}{\Gamma, \exists w\, \psi \Rightarrow \Delta}\ (\exists \Rightarrow)$$

where $z$ is free for $w$ in $\psi$, $z$ is not free in $\Gamma \cup \Delta \cup \{\forall w \psi\}$, and $\mathbf{t}$ is any term free for $w$ in $\psi$.

(3) Consider the bounded existential and universal $(2,1)$-ary quantifiers $\overline{\forall}$ and $\overline{\exists}$ (corresponding to $\forall x.p_1(x) \rightarrow p_2(x)$ and $\exists x.p_1(x) \wedge p_2(x)$ used in syllogistic reasoning). Their corresponding rules can be formulated as follows:

$$\{p_2(c) \Rightarrow\ ,\ \Rightarrow p_1(c)\}/\overline{\forall} v_1\ (p_1(v_1), p_2(v_1)) \Rightarrow$$
$$\{p_1(v_1) \Rightarrow p_2(v_1)\}/ \Rightarrow \overline{\forall} v_1\ (p_1(v_1), p_2(v_1))$$
$$\{p_1(v_1), p_2(v_1) \Rightarrow\}/\overline{\exists}\ v_1(p_1(v_1), p_2(v_1)) \Rightarrow$$
$$\{\Rightarrow p_1(c)\ ,\ \Rightarrow p_2(c)\}/ \Rightarrow \overline{\exists} v_1(p_1(v_1), p_2(v_1))$$

Applications of these rules are of the form:

$$\frac{\Gamma, \psi_2\{\mathbf{t}/z\} \Rightarrow \Delta \quad \Gamma \Rightarrow \psi_1\{\mathbf{t}/z\}, \Delta}{\Gamma, \overline{\forall} z\ (\psi_1, \psi_2) \Rightarrow \Delta} \qquad \frac{\Gamma, \psi_1\{y/z\} \Rightarrow \psi_2\{y/z\}, \Delta}{\Gamma \Rightarrow \overline{\forall} z\ (\psi_1, \psi_2), \Delta}$$

$$\frac{\Gamma, \psi_1\{y/z\}, \psi_2\{y/z\} \Rightarrow \Delta}{\Gamma, \overline{\exists} z\ (\psi_1, \psi_2) \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \psi_1\{\mathbf{t}/x\}, \Delta \quad \Gamma \Rightarrow \psi_2\{\mathbf{t}/x\}, \Delta}{\Gamma \Rightarrow \overline{\exists} z\ (\psi_1, \psi_2), \Delta}$$

where $\mathbf{t}$ and $y$ are free for $z$ in $\psi_1$ and $\psi_2$, $y$ does not occur free in $\Gamma \cup \Delta \cup \{\overline{\exists} z(\psi_1, \psi_2)\}$.

(4) Consider the (2,2)-ary rule

$$\{p_1(v_1,v_2) \Rightarrow \ ,\ p_1(v_3,d) \Rightarrow p_2(c,d)\}/ \Rightarrow \mathcal{Q}v_1v_2(p_1(v_1,v_2),p_2(v_1,v_2))$$

Its application is of the form:

$$\frac{\Gamma,\psi_1\{w_1/z_1,w_2/z_2\} \Rightarrow \Delta \quad \Gamma,\psi_1\{w_3/z_1,\mathbf{t}_1/z_2\} \Rightarrow \Delta,\psi_2\{\mathbf{t}_2/z_1,\mathbf{t}_1/z_2\}}{\Gamma \Rightarrow \Delta, \mathcal{Q}z_1z_2(\psi_1,\psi_2)}$$

where $w_1, w_2, w_3, \mathbf{t}_1, \mathbf{t}_2$ satisfy the appropriate conditions.

Note that although the derivability of the $\alpha$-axiom is essential for any logical system, it is not guaranteed to be derivable in a canonical system. What natural syntactic conditions guarantee its derivability is still a question for further research. For now we explicitly add the $\alpha$-axiom to the canonical calculi.

*Notation:* (Following [2], notations 3-5.) Let $-t = f, -f = t$ and $ite(t,A,B) = A$, $ite(f,A,B) = B$. Let $\Phi, A^s$ (where $\Phi$ may be empty) denote $ite(s,\Phi \cup \{A\},\Phi)$. For instance, the sequents $A \Rightarrow$ and $\Rightarrow A$ are denoted by $A^{-s} \Rightarrow A^s$ for $s = f$ and $s = t$ respectively. According to this notation, a $(n,k)$-ary canonical rule is of the form:

$$\{\Sigma_j \Rightarrow \Pi_j\}_{1\leq j\leq m}/\mathcal{Q}\overrightarrow{v}(p_1(\overrightarrow{v}),...,p_n(\overrightarrow{v}))^{-s} \Rightarrow \mathcal{Q}\overrightarrow{v}(p_1(\overrightarrow{v}),...,p_n(\overrightarrow{v}))^s$$

for $s \in \{t,f\}$. For further abbreviation, we denote such rule by $\{\Sigma_j \Rightarrow \Pi_j\}_{1\leq j\leq m}/\mathcal{Q}(s)$.

**Definition 2.6.** A Gentzen-type calculus $G$ is *canonical* if in addition to the $\alpha$-axiom $A \Rightarrow A'$ for $A \equiv_\alpha A'$ and the standard structural rules, $G$ has only canonical rules.

**Definition 2.7.** Two $(n,k)$-ary canonical introduction rules $\Theta_1/C_1$ and $\Theta_2/C_2$ for $\mathcal{Q}$ are *dual* if for some $s \in \{t,f\}$: $C_1 = A^{-s} \Rightarrow A^s$ and $C_2 = A^s \Rightarrow A^{-s}$, where $A = \mathcal{Q}v_1...v_k(p_1(v_1,...,v_k),...,p_n(v_1,...,v_k))$.

Although we can define arbitrary canonical systems using our simplified language $L_k^n$, our quest is for systems, the syntactic rules of which define the semantic meaning of logical connectives/quantifiers. Thus we are interested in calculi with a “reasonable” or “non-contradictory” set of rules, which allows for defining a sound and complete semantics for the system. This can be captured syntactically by the following extension of the *coherence* criterion of [2, 23].

**Definition 2.8.** For two sets of clauses $\Theta_1, \Theta_2$ over $L_k^n$, $\mathsf{Rnm}(\Theta_1 \cup \Theta_2)$ is a set $\Theta_1 \cup \Theta_2'$, where $\Theta_2'$ is obtained from $\Theta_2$ by a fresh renaming of constants and variables which occur in $\Theta_1$.

Henceforth it will be convenient (but not essential) to assume that the fresh constants used for the renaming are in $L$.

**Definition 2.9. (Coherence)**[3] A canonical calculus $G$ is *coherent* if for every two dual canonical rules $\Theta_1/ \Rightarrow A$ and $\Theta_2/A \Rightarrow$, the set of clauses $\mathsf{Rnm}(\Theta_1 \cup \Theta_2)$ is classically inconsistent.

[3] A strongly related coherence criterion is defined in [17], where linear logic is used to reason about various sequent systems. Our coherence criterion is also equivalent in the context of canonical calculi to the reductivity condition of [?, 8], as will be explained in the sequel.

Note that the principle of renaming of clashing constants and variables is similar to the one used in first-order resolution. The importance of this principle for the definition of coherence will be explained in the sequel.

**Proposition 2.10. (Decidability of coherence)** *The coherence of a canonical calculus G is decidable.*

*Proof.* The question of classical consistency of a finite set of clauses without function symbols (over $L_k^n$) can be shown to be equivalent to satisfiability of a finite set of universal formulas with no function symbols. This is decidable (by an obvious application of Herbrand's theorem).

## 3. The semantic framework

3.1. **Non-deterministic matrices.** Our main semantic tool are non-deterministic matrices (Nmatrices), first introduced in [2, 3] and extended in [22, 23]. These structures are a generalization of the standard concept of a many-valued matrix, in which the truth-value of a formula is chosen non-deterministically from a given non-empty set of truth-values. Thus, given a set of truth-values $\mathcal{V}$, we can generalize the notion of a distribution function of an $(n, k)$-ary quantifier $\mathcal{Q}$ (from Definition. 1.1) to a function $\lambda_{\mathcal{Q}} : P^+(\mathcal{V}^n) \to P^+(\mathcal{V})$. In other words, given some distribution $Y$ of n-ary vectors of truth values, the interpretation function non-deterministically chooses the truth value assigned to $\mathcal{Q}\overrightarrow{z}(\psi_1, ..., \psi_n)$ out from $\lambda_{\mathcal{Q}}[Y]$ .

**Definition 3.1. (Non-deterministic matrix)** A non-deterministic matrix (henceforth Nmatrix) for $L$ is a tuple $\mathcal{M} =< \mathcal{V}, \mathcal{G}, \mathcal{O} >$, where:

- $\mathcal{V}$ is a non-empty set of truth values.
- $\mathcal{G}$ (designated truth values) is a non-empty proper subset of $\mathcal{V}$.
- $\mathcal{O}$ is a set of interpretation functions: for every $(n, k)$-ary quantifier $\mathcal{Q}$ of L, $\mathcal{O}$ includes the corresponding distribution function $\tilde{\mathcal{Q}}_{\mathcal{M}} : P^+(\mathcal{V}^n) \to P^+(\mathcal{V})$.

Note the special treatment of propositional connectives in the definition above. In [2, 23], an Nmatrix includes an interpretation function $\tilde{\diamond} : \mathcal{V}^n \to P^+(\mathcal{V})$ for every $n$-ary connective of the language; given a valuation $v$, the truth value $v[\diamond(\psi_1, ..., \psi_n)]$ is chosen non-deterministically from $\tilde{\diamond}[\langle v[\psi_1], ..., v[\psi_n]\rangle]$. In the definition above, the interpretation of a propositional connective $\diamond$ is a function of another type: $\tilde{\diamond} : P^+(\mathcal{V}^n) \to P^+(\mathcal{V})$. This can be thought as a generalization of the previous definition, identifying the tuple $\langle v[\psi_1], ..., v[\psi_n]\rangle$ with the singleton $\{\langle v[\psi_1], ..., v[\psi_n]\rangle\}$. The advantage of this generalization is that it allows for a uniform treatment of both quantifiers and propositional connectives.

**Definition 3.2. (L-structure)** Let $\mathcal{M}$ be an Nmatrix for $L$. An L-structure for $\mathcal{M}$ is a pair $S = \langle D, I\rangle$ where $D$ is a (non-empty) domain and $I$ is a function interpreting constants, predicate symbols and function symbols of $L$, satisfying the following conditions: $I[c] \in D$, $I[p^n] : D^n \to \mathcal{V}$ is an n-ary predicate, and $I[f^n] : D^n \to D$ is an n-ary function.
$I$ is extended to interpret closed terms of $L$ as follows:

$$I[f(\mathbf{t}_1, ..., \mathbf{t}_n)] = I[f][I[\mathbf{t}_1], ..., I[\mathbf{t}_n]]$$

Here a note on our treatment of quantification in the framework of Nmatrices is in order. The standard approach to interpreting quantified formulas is by using *objectual* (or referential) semantics, where the variable is thought of as ranging over a set of objects from the domain (see. e.g. [10, 11]). An alternative approach is *substitutional* quantification ([16]), where quantifiers are interpreted substitutionally, i.e. a universal (an existensial) quantification is true if and only if every one (at least one) of its substitution instances is true (see. e.g. [20, 9]). [22] explains the motivation behind choosing the substitutional approach for the framework of Nmatrices, and points out the problems of the objectual approach in this context. The substitutional approach assumes that every element of the domain has a closed term referring to it. Thus given a structure $S = \langle D, I\rangle$, we extend the language $L$ with *individual constants*, one for each element of $D$.

**Definition 3.3. ( L(D) )** Let S=$\langle D, I\rangle$ be an $L$-structure for an Nmatrix $\mathcal{M}$. $L(D)$ is the language obtained from $L$ by adding to it the set of *individual constants* $\{\overline{a} \mid a \in D\}$. $S' = \langle D, I'\rangle$ is the $L(D)$-structure, such that $I'$ is an extension of $I$ satisfying: $I'[\overline{a}] = a$.

Given an $L$-structure $S = \langle D, I\rangle$, we shall refer to the extended $L(D)$-structure $\langle D, I'\rangle$ as $S$ and to $I'$ as $I$ when the meaning is clear from the context.

**Definition 3.4. ($S$-substitution)** Given an $L$-structure $S = \langle D, I\rangle$ for an Nmatrix $\mathcal{M}$ for $L$, an *$S$-substitution* is a function $\sigma : Var \to Trm^{cl}_{L(D)}$. It is extended to $\sigma : Trm_L \cup Frm_L \to Trm^{cl}_{L(D)} \cup Frm^{cl}_{L(D)}$ as follows: for a term $\mathbf{t}$ of $L(D)$, $\sigma[\mathbf{t}]$ is the closed term obtained from $\mathbf{t}$ by replacing every $x \in Fv[\mathbf{t}]$ by $\sigma[x]$. For a formula $\varphi$, $\sigma[\varphi]$ is the sentence obtained from $\varphi$ by replacing every $x \in Fv[\varphi]$ by $\sigma[x]$.
Given a set $\Gamma$ of formulas, we denote the set $\{\sigma[\psi] \mid \psi \in \Gamma\}$ by $\sigma[\Gamma]$.

The motivation for the following definition is purely technical and is related to extending the language with the set of individual constants $\{\overline{a} \mid a \in D\}$. Suppose we have a closed term $\mathbf{t}$, such that $I[\mathbf{t}] = a \in D$. But $a$ also has an individual constant $\overline{a}$ referring to it. We would like to be able to substitute $\mathbf{t}$ for $\overline{a}$ in every context.

**Definition 3.5. (Congruence of terms and formulas)** Let $S$ be an $L$-structure for an Nmatrix $\mathcal{M}$.
The relation $\sim^S$ between terms of $L(D)$ is defined inductively as follows:

- $x \sim^S x$
- For closed terms $\mathbf{t}, \mathbf{t}'$ of $L(D)$: $\mathbf{t} \sim^S \mathbf{t}'$ when $I[\mathbf{t}] = I[\mathbf{t}']$.
- If $\mathbf{t}_1 \sim^S \mathbf{t}'_1, ..., \mathbf{t}_n \sim^S \mathbf{t}'_n$, then $f(\mathbf{t}_1, ..., \mathbf{t}_n) \sim^S f(\mathbf{t}'_1, ..., \mathbf{t}'_n)$.

The relation $\sim^S$ between formulas of $L(D)$ is defined as follows:

- If $\mathbf{t}_1 \sim^S \mathbf{t}'_1, \mathbf{t}_2 \sim^S \mathbf{t}'_2, ..., \mathbf{t}_n \sim^S \mathbf{t}'_n$, then $p(\mathbf{t}_1, ..., \mathbf{t}_n) \sim^S p(\mathbf{t}'_1, ..., \mathbf{t}'_n)$.
- If $\psi_1\{\overrightarrow{z}/\overrightarrow{x}\} \sim^S \varphi_1\{\overrightarrow{z}/\overrightarrow{y}\}, ..., \psi_n\{\overrightarrow{z}/\overrightarrow{x}\} \sim^S \varphi_n\{\overrightarrow{z}/\overrightarrow{y}\}$, where $\overrightarrow{x} = x_1...x_k$ and $\overrightarrow{y} = y_1...y_k$ are distinct variables and $\overrightarrow{z} = z_1...z_k$ are new distinct variables, then $\mathcal{Q}\overrightarrow{x}(\psi_1, ..., \psi_n) \sim^S \mathcal{Q}\overrightarrow{y}(\varphi_1, ..., \varphi_n)$ for any $(n, k)$-ary quantifier $\mathcal{Q}$ of $L$.

Intuitively, $\psi \sim^S \psi'$ if $\psi'$ can be obtained from $\psi$ by possibly renaming bound variables and by any number of substitutions of a closed term $\mathbf{t}$ for another closed term $\mathbf{s}$, so that $I[\mathbf{t}] = I[\mathbf{s}]$.

**Lemma 3.6.** *([22]) Let $S$ be an $L$-structure for an Nmatrix $\mathcal{M}$. Let $\psi, \psi'$ be formulas of $L(D)$. Let $\mathbf{t}, \mathbf{t}'$ be closed terms of $L(D)$, such that $\mathbf{t} \sim^S \mathbf{t}'$.*

(1) *If* $\psi \equiv_\alpha \psi'$*, then* $\psi \sim^S \psi'$.
(2) *If* $\psi \sim^S \psi'$*, then* $\psi\{\boldsymbol{t}/x\} \sim^S \psi'\{\boldsymbol{t}'/x\}$.

**Definition 3.7. (Legal valuation)** Let $S = \langle D, I\rangle$ be an $L$-structure for an Nmatrix $\mathcal{M}$. An $S$-valuation $v : Frm^{cl}_{L(D)} \to \mathcal{V}$ is *legal in* $\mathcal{M}$ if it satisfies the following conditions:

(1) $v[\psi] = v[\psi']$ for every two sentences $\psi, \psi'$ of $L(D)$, such that $\psi \sim^S \psi'$.
(2) $v[p(\mathbf{t}_1, ..., \mathbf{t}_n)] = I[p][I[\mathbf{t}_1], ..., I[\mathbf{t}_n]]$.
(3) For every $(n, k)$-ary quantifier $\mathcal{Q}$ of $L$, $v[\mathcal{Q}x_1, ..., x_k(\psi_1, ..., \psi_n)]$ should be an element of $\tilde{\mathcal{Q}}_{\mathcal{M}}[\{\langle v[\psi_1\{\overline{a}_1/x_1, ..., \overline{a}_k/x_k\}], ..., v[\psi_n\{\overline{a}_1/x_1, ..., \overline{a}_k/x_k\}]\rangle \mid a_1, ..., a_k \in D\}]$.

Note that in case $\mathcal{Q}$ is a propositional connective (for $k = 0$), the function $\tilde{\mathcal{Q}}_{\mathcal{M}}$ is applied to a singleton, as was explained above.

*Notation:* For a set of sequents $\mathcal{S}$, we shall write $\mathcal{S} \vdash_G \Gamma \Rightarrow \Delta$ if a sequent $\Gamma \Rightarrow \Delta$ has a proof from $\mathcal{S}$ in $G$.

**Definition 3.8.** Let $S = \langle D, I\rangle$ be an $L$-structure for an Nmatrix $\mathcal{M}$.

(1) An $\mathcal{M}$-legal $S$-valuation $v$ is *a model of a sentence* $\psi$ *in* $\mathcal{M}$, denoted by $S, v \models_{\mathcal{M}} \psi$, if $v[\psi] \in \mathcal{G}$.
(2) Let $v$ be an $\mathcal{M}$-legal $S$-valuation. A sequent $\Gamma \Rightarrow \Delta$ is $\mathcal{M}$*-valid in* $\langle S, v\rangle$ if for every $S$-substitution $\sigma$: if $S, v \models_{\mathcal{M}} \sigma[\psi]$ for every $\psi \in \Gamma$, then there is some $\varphi \in \Delta$, such that $S, v \models_{\mathcal{M}} \sigma[\varphi]$.
(3) A sequent $\Gamma \Rightarrow \Delta$ is $\mathcal{M}$-valid, denoted by $\vdash_{\mathcal{M}} \Gamma \Rightarrow \Delta$, if for every $L$-structure $S$ and every $\mathcal{M}$-legal $S$-valuation $v$, $\Gamma \Rightarrow \Delta$ is $\mathcal{M}$-valid in $\langle S, v\rangle$.
(4) For a set of sequents $\mathcal{S}$, $\mathcal{S} \vdash_{\mathcal{M}} \Gamma \Rightarrow \Delta$ if for every $L$-structure $S$ and every $\mathcal{M}$-legal $S$-valuation $v$: whenever the sequents of $\mathcal{S}$ are $\mathcal{M}$-valid in $\langle S, v\rangle$, $\Gamma \Rightarrow \Delta$ is also $\mathcal{M}$-valid in $\langle S, v\rangle$.

**Definition 3.9.** A system $G$ is *strongly sound*[4] for an Nmatrix $\mathcal{M}$ if for every set $\mathcal{S}$ of sequents closed under substitution: $\mathcal{S} \vdash_G \Gamma \Rightarrow \Delta$ entails $\mathcal{S} \vdash_{\mathcal{M}} \Gamma \Rightarrow \Delta$. A system $G$ is *strongly complete* for an Nmatrix $\mathcal{M}$ if for every set $\mathcal{S}$ of sequents closed under substitution: $\mathcal{S} \vdash_{\mathcal{M}} \Gamma \Rightarrow \Delta$ entails $\mathcal{S} \vdash_G \Gamma \Rightarrow \Delta$. An Nmatrix $\mathcal{M}$ is *strongly characteristic* for $G$ if $G$ is strongly sound and strongly complete for $\mathcal{M}$.

Note that since the empty set of sequents is closed under substitutions, strong soundness implies (weak) soundness[5]. A similar remark applies to completeness and a characteristic Nmatrix.

3.2. **Semantics for simplified languages $L^n_k$.** In addition to $L$-structures for languages with $(n, k)$-ary quantifiers, we also use $L^n_k$-structures for the simplified languages $L^n_k$, used for formulating the canonical rules. To make the distinction clearer, we shall use the metavariable $S$ for the former and $\mathcal{N}$ for the latter. Since the formulas of $L^n_k$ are always atomic, the specific 2Nmatrix for which $\mathcal{N}$ is defined is immaterial, and can be omitted. We may even speak of classical validity of sequents over $L^n_k$. Thus henceforth instead of speaking of $\mathcal{M}$-validity of a set of clauses $\Theta$ over $L^n_k$, we will speak simply of validity.

[4]A more general definition would be without the restriction concerning the closure of $\mathcal{S}$ under substitution. However, in this case we would need to add substitution as a structural rule to canonical calculi.

[5]A system $G$ is (weakly) sound for an Nmatrix $\mathcal{M}$ if $\vdash_G \Gamma \Rightarrow \Delta$ entails $\vdash_{\mathcal{M}} \Gamma \Rightarrow \Delta$.

Next we define the notion of a *distribution* of $L_k^n$-structures.

**Definition 3.10.** Let $\mathcal{N} = \langle D, I\rangle$ be a structure for $L_k^n$. $Dist_{\mathcal{N}}$, the distribution of $\mathcal{N}$ is defined as follows:

$$Dist_{\mathcal{N}} = \{\langle I[p_1][a_1, ..., a_k], ..., I[p_n][a_1, ..., a_k]\rangle \mid a_1, ..., a_k \in D\}$$

We say that an $L_k^n$-structure $\mathcal{N}$ is $\mathcal{E}$-characteristic if $Dist_{\mathcal{N}} = \mathcal{E}$.

Note that the distribution of an $\mathcal{L}_0^n$-structure $\mathcal{N}$ is $Dist_{\mathcal{N}} = \{\langle I[p_1], ..., I[p_n]\rangle\}$ and so it is always a singleton. Furthermore, the validity of a set of clauses over $\mathcal{L}_0^n$ can be reduced to propositional satisfiability as stated in the following lemma which can be easily proved:

**Lemma 3.11.** *Let $\mathcal{N}$ be a $\mathcal{L}_0^n$-structure. Assume that $Dist_{\mathcal{N}} = \{\langle s_1, ..., s_n\rangle\}$ for some $s_1, ..., s_n \in \{t, f\}$. Let $v_{Dist_{\mathcal{N}}}$ be any propositional valuation satisfying $v[p_i] = s_i$ for every $1 \le i \le n$. A set of clauses $\Theta$ is valid in $\mathcal{N}$ iff $v_{Dist_{\mathcal{N}}}$ propositionally satisfies $\Theta$.*

Now we turn to the case $k = 1$. In this case it is convenient to define a special kind of $\mathcal{L}_1^n$-structures which we call *canonical* structures. These structures are sufficient to reflect the behavior of all possible $\mathcal{L}_1^n$-structures.

**Definition 3.12.** Let $\mathcal{E} \in P^+(\{t, f\}^n)$. A $\mathcal{L}_1^n$-structure $\mathcal{N} = \langle D, I\rangle$ is $\mathcal{E}$-canonical if $D = \mathcal{E}$ and for every $b = \langle s_1, ..., s_n\rangle \in D$ and every $1 \le i \le n$: $I[p_i][b] = s_i$.

Clearly, every $\mathcal{E}$-canonical $\mathcal{L}_1^n$-structure is $\mathcal{E}$-characteristic.

**Lemma 3.13.** *Let $\Theta$ be a set of clauses over $\mathcal{L}_1^n$, which is valid in some structure $\mathcal{N} = \langle D, I\rangle$. Then there exists a $Dist_{\mathcal{N}}$-canonical structure $\mathcal{N}'$ in which $\Theta$ is valid.*

*Proof.* Suppose that $\Theta$ is valid in a structure $\mathcal{N} = \langle D, I\rangle$. Define the $\mathcal{L}_1^n$-structure $\mathcal{N}' = \langle I', D'\rangle$ as follows:

- $D' = Dist_{\mathcal{N}}$.
- $I'[c] = \langle I[p_1][I[c]], ..., I[p_n][I[c]]\rangle$ for every constant $c$ occurring in $\Theta$.
- For every $1 \le i \le n$: $I'[p_i][\langle s_1, ..., s_n\rangle] = t$ iff $s_i = t$.

Clearly, $\mathcal{N}'$ is $Dist_{\mathcal{N}}$-canonical. It is easy to verify that $\Theta$ is valid in $\mathcal{N}'$. □

**Corollary 3.14.** *Let $\mathcal{E} \in P^+(\{t, f\}^n)$. For a finite set of clauses $\Theta$ over $\mathcal{L}_1^n$, the question whether $\Theta$ is valid in a $\mathcal{E}$-characteristic structure is decidable.*

*Proof.* Follows from lemma 3.13 and the fact that for any $\mathcal{E} \in P^+(\{t, f\}^n)$, there are finitely many $\mathcal{E}$-canonical structures to check.

## 4. Canonical systems with (n,k)-ary quantifiers for $k \in \{0, 1\}$

Now we turn to the class of canonical systems with $(n, k)$-ary quantifiers for the case of $k \in \{0, 1\}$ and $n \ge 1$. Henceforth, unless stated otherwise, we assume that $k \in \{0, 1\}$.

4.1. **Semantics for canonical systems for $k \in \{0,1\}$.** In this section we explore the connection between the coherence of a canonical calculus $G$, the existence for it of a strongly characteristic 2Nmatrix, and strong cut-elimination (in a sense explained below.) We start by defining the notion of *suitability* for $G$.

**Definition 4.1. (Suitability for $G$)** Let $G$ be a canonical calculus over $L$. A 2Nmatrix $\mathcal{M}$ is *suitable* for $G$ if for every $(n,k)$-ary canonical rule $\Theta/\mathcal{Q}(s)$ of $G$ (where $s \in \{t,f\}$), it holds that for every $L_k^n$-structure $\mathcal{N}$ in which $\Theta$ is valid: $\tilde{\mathcal{Q}}_{\mathcal{M}}[Dist_{\mathcal{N}}] = \{s\}$.

Next we prove that if a 2Nmatrix $\mathcal{M}$ is suitable for $G$, then $G$ is strongly sound for $\mathcal{M}$.

**Theorem 4.2.** *Let $G$ be a canonical calculus and $\mathcal{M}$ - a 2Nmatrix suitable for $G$. Then $G$ is strongly sound for $\mathcal{M}$.*

*Proof.* Suppose that $\mathcal{M}$ is suitable for $G$. Let $S = \langle D, I\rangle$ be some $L$-structure and $v$ - an $\mathcal{M}$-legal $S$-valuation. Let $\mathcal{S}$ be any set of sequents closed under substitution. We will show that if the sequents of $\mathcal{S}$ are $\mathcal{M}$-valid in $\langle S, v\rangle$, then any sequent provable from $\mathcal{S}$ in $G$ is $\mathcal{M}$-valid in $\langle S, v\rangle$. Obviously, the axioms of $G$ are $\mathcal{M}$-valid, and the structural rules, including cut, are strongly sound. It remains to show that for every application of a canonical rule $R$ of $G$: if the premises of $R$ are $\mathcal{M}$-valid in $\langle S, v\rangle$, then its conclusion is $\mathcal{M}$-valid in $\langle S, v\rangle$. We will show this for the case of $k = 1$, leaving the easier case of $k = 0$ to the reader.
Let $R$ be an $(n,1)$-ary rule of $G$:

$$R = \Theta_R/\mathcal{Q}v_1(p_1(v_1),...,p_n(v_1))^{-r} \Rightarrow \mathcal{Q}v_1(p_1(v_1),...,p_n(v_1))^r$$

where $r \in \{t,f\}$ and $\Theta_R = \{\Sigma_j \Rightarrow \Pi_j\}_{1\leq j\leq m}$. An application of $R$ is of the form:

$$\frac{\{\Gamma, \chi[\Sigma_j] \Rightarrow \chi[\Pi_j], \Delta\}_{1\leq j\leq m}}{\Gamma, \mathcal{Q}z(\chi[p_1],...,\chi[p_n])^{-r} \Rightarrow \Delta, \mathcal{Q}z(\chi[p_1],...,\chi[p_n])^r}$$

where $\chi$ is some $\langle R, \Gamma\cup\Delta, z\rangle$-mapping. Suppose that $\{\Gamma, \chi[\Sigma_j] \Rightarrow \chi[\Pi_j], \Delta\}_{1\leq j\leq m}$ is $\mathcal{M}$-valid in $\langle S, v\rangle$. We will now show that $\Gamma, \mathcal{Q}z(\chi[p_1],...,\chi[p_n])^{-r} \Rightarrow \Delta, \mathcal{Q}z(\chi[p_1],...,\chi[p_n])^r$ is also $\mathcal{M}$-valid in $\langle S, v\rangle$.

(a) Let $\sigma$ be an $S$-substitution, such that $S, v \models_{\mathcal{M}} \sigma[\Gamma]$ and for every $\psi \in \Delta$: $S, v \not\models_{\mathcal{M}} \sigma[\psi]$.

Denote by $\widetilde{\psi}$ the $L$-formula obtained from a formula $\psi$ by substituting every free occurrence of $w \in Fv[\psi] - \{z\}$ for $\sigma[w]$.
Let $\mathcal{E} = \{\langle v[\widetilde{\chi[p_1]}\{\overline{a}/z\}],...,v[\widetilde{\chi[p_n]}\{\overline{a}/z\}]\rangle \mid a \in D\}$. We will show that $\tilde{\mathcal{Q}}[\mathcal{E}] = \{r\}$, and so $v[\sigma[Qz(\chi[p_1],...,\chi[p_n])]] = r$. From (a) it will follow that $\Gamma, \mathcal{Q}z(\chi[p_1],...,\chi[p_n])^{-r} \Rightarrow \Delta, \mathcal{Q}z(\chi[p_1],...,\chi[p_n])^r$ is $\mathcal{M}$-valid in $\langle S, v\rangle$.
We prove this by showing that $\Theta_R$ is valid in some $\mathcal{E}$-characteristic $L_k^n$-structure. Then, by suitability of $\mathcal{M}$, we shall conclude that $\tilde{\mathcal{Q}}_{\mathcal{M}}[\mathcal{E}] = r$.
Construct the $L_k^n$-structure $\mathcal{N} = \langle D', I'\rangle$ as follows:

- $D' = D$.
- For every $a \in D$: $I'[p_i][a] = v[\widetilde{\chi[p_i]}\{\overline{a}/z\}]$.
- For every constant $c$, $I'[c] = I[\sigma[\chi[c]]]$.

We will now show that $\Theta_R = \{\Sigma_j \Rightarrow \Pi_j\}_{1\leq j\leq m}$ is valid in $\mathcal{N}$. Suppose for contradiction that it is not so. Then there exists some $1 \leq j \leq m$, for which $\Sigma_j \Rightarrow \Pi_j$ is not valid in $\mathcal{N}$. Thus there is some $\mathcal{N}$-substitution $\eta$, such that:

(b) whenever $p_i(\mathbf{t}) \in \Pi_j \cup \Sigma_j$: $p_i(\mathbf{t}) \in ite(I'[p_i][I'[\eta[\mathbf{t}]]], \Sigma_j, \Pi_j)$.

We show now that $\Gamma, \chi[\Sigma_j] \Rightarrow \chi[\Pi_j], \Delta$ is not $\mathcal{M}$-valid in $\langle S, v\rangle$, in contradiction to our assumption about the premises of the above application.
Let $\psi \in ite(s, \chi[\Sigma_j], \chi[\Pi_j])$ for $s \in \{t, f\}$. Let $\sigma'$ be the $S$-substitution similar to $\sigma$ except that $\sigma'[\chi[y]] = \overline{a}_y$, where $a_y = I'[\eta[y]]$ for every variable $y$ occurring in $\Theta_R$. Note that $\sigma'$ is well-defined, since for every two different variables $x, y$: $\chi[x] \neq \chi[y]$ (recall defn. 2.3). Then one of the following holds:

- $\psi = \chi[p_i]\{\chi[c]/z\}$, where $p_i(c) \in ite(s, \Sigma_j, \Pi_j)$ and $\chi[c]$ is some term free for $z$ in $\chi[p_i]$, such that for any variable $y$ occurring in $\Theta_R$, $\chi[y]$ does not occur in $\chi[c]$. Recall that by (b), $I'[p_i][I'[\eta[c]]] = s$. And so:

$$v[\sigma'[\psi]] = v[\sigma'[\chi[p_i]\{\chi[c]/z\}]] = v[\widetilde{\chi[p_i]}\{\sigma'[\chi[c]]/z\}] = v[\widetilde{\chi[p_i]}\{\sigma[\chi[c]]/z\}]$$

  (Recall that for every variable $y$ occurring in $\Theta_R$, $\chi[y]$ does not occur free in $\mathcal{Q}z(\chi[p_1], ..., \chi[p_n])$, and $\sigma, \sigma'$ only differ for variables $\chi[z]$ where $z$ occurs in $\Theta_R$.) By lemma 3.6-2 and the legality of $v$:

$$v[\widetilde{\chi[p_i]}\{\sigma[\chi[c]]/z\}] = v[\widetilde{\chi[p_i]}\{\overline{I[\sigma[\chi[c]]]}/z\}]$$

  By definition of $I'$, $I'[c] = I[\sigma[\chi[c]]]$ and so:

$$v[\widetilde{\chi[p_i]}\{\overline{I[\sigma[\chi[c]]]}/z\}] = v[\widetilde{\chi[p_i]}\{\overline{I'[c]}/z\}] = I'[p_i][I'[c]] = I'[p_i][I'[\eta[c]]] = s$$

- $\psi = \chi[p_i]\{\chi[y]/z\}$, where $p_i(y) \in ite(s, \Sigma_j, \Pi_j)$ and $\chi[y]$ does not occur in $\Gamma \cup \Delta \cup \{\mathcal{Q}z(\psi_1, ..., \psi_n)\}$ and is free for $z$ in $\chi[p_i]$. Then $I'[p_i][I'[\eta[y]]] = s$.
  Let $a = I'[\eta[y]]$. Then, $\sigma'[\chi[y]] = \overline{a}$ and so:

$$v[\sigma'[\psi]] = v[\sigma'[\chi[p_i]\{\chi[y]/z\}] = v[\widetilde{\chi[p_i]}\{\sigma'[\chi[y]]/z\}] =$$
$$= v[\widetilde{\chi[p_i]}\{\overline{a}/z\}] = I'[p_i][a] = I'[p_i][I'[\mu[y]]] = s$$

Thus we have shown that $v[\sigma'[\psi]] = s$ whenever $\psi \in ite(s, \chi[\Sigma_j], \chi[\Pi_j])$. Also, there is no variable $y$ occurring in $\Theta_R$, such that $\chi[y]$ occurs in $\Gamma \cup \Delta$, and so $\sigma[\Gamma] = \sigma'[\Gamma]$ and $\sigma[\Delta] = \sigma'[\Delta]$. Thus for every $\psi \in \Gamma \cup \chi[\Sigma_j]$, $v[\sigma'[\psi]] = t$ while for every $\varphi \in \Delta \cup \chi[\Pi_j]$, $v[\sigma'[\varphi]] = f$. Hence, $\Gamma, \chi[\Sigma_j] \Rightarrow \Delta, \chi[\Pi_j]$ is not $\mathcal{M}$-valid in $\langle S, v\rangle$, in contradiction to our assumption on the validity of the premises of the application above.
We have shown that $\{\Sigma_j \Rightarrow \Pi_j\}_{1\leq j\leq m}$ is valid in $\mathcal{N}$. Obviously[6], $Dist_{\mathcal{N}} = \mathcal{E}$. Since $\mathcal{M}$ is suitable for $G$: $\tilde{\mathcal{Q}}_{\mathcal{M}}[\mathcal{E}] = \{r\}$ and so $v[\sigma[\mathcal{Q}z(\chi[p_1], ..., \chi[p_n])]] = r$. From this fact and assumption (a) it follows that $\Gamma, \mathcal{Q}z(\chi[p_1], ..., \chi[p_n])^{-r} \Rightarrow \Delta, \mathcal{Q}z(\chi[p_1], ..., \chi[p_n])^r$ is $\mathcal{M}$-valid in $\langle S, v\rangle$. □

Now we come to the construction of a characteristic 2Nmatrix for every coherent canonical calculus.

[6]Recall that $\mathcal{E} = \{\langle v[\widetilde{\chi[p_1]}\{\overline{a}/z\}], ..., v[\widetilde{\chi[p_n]}\{\overline{a}/z\}]\rangle \mid a \in D\}$ and $I'[p_i][a] = v[\widetilde{\chi[p_i]}\{\overline{a}/z\}]$ for every $a \in D$ and every $1 \leq i \leq n$.

**Definition 4.3.** Let $G$ be a coherent canonical calculus. The Nmatrix $\mathcal{M}_G$ for $L$ is defined as follows for every $(n,k)$-ary quantifier $\mathcal{Q}$ of $L$, every $s \in \{t,f\}$ and every $\mathcal{E} \in P^+(\{t,f\}^n)$):

$$\tilde{\mathcal{Q}}_{\mathcal{M}_G}[\mathcal{E}] = \begin{cases} \{s\} & \textit{if } \Theta/\mathcal{Q}(s) \in G \textit{ and} \\ & \Theta \textit{ is valid in some } \mathcal{E}-\textit{canonical } L^n_k - \textit{structure} \\ \{t,f\} & \textit{otherwise} \end{cases}$$

First of all, note that by corollary 3.14, the above definition is constructive. Next, let us show that $\mathcal{M}_G$ is well-defined. Assume by contradiction that there are two dual rules $\Theta_1/\Rightarrow A$ and $\Theta_2/A\Rightarrow$, such that both $\Theta_1$ and $\Theta_2$ are valid in some $\mathcal{E}$-canonical structures $\mathcal{N}_1, \mathcal{N}_2$ respectively. Obtain $\Theta_2'$ from $\Theta_2$ by renaming of constants and variables which occur in $\Theta_1$. Then clearly $\Theta_2'$ is also valid in some $\mathcal{E}$-canonical structure $\mathcal{N}_3$. If $k=0$, by lemma 3.11, the set of clauses $\Theta_1 \cup \Theta_2'$ is satisfiable by a (classical) propositional valuation $v_{\mathcal{E}}$ and is thus classically consistent, in contradiction to the coherence of $G$ (see defn. 2.9).
Otherwise, $k=1$. The only difference between different $\mathcal{E}$-canonical structures is in the interpretation of constants, and since the sets of constants occurring in $\Theta_1$ and $\Theta_2'$ are disjoint, an $\mathcal{E}$-canonical structure $\mathcal{N}' = \langle D', I'\rangle$ (for the extended language containing the constants of both $\Theta_1$ and $\Theta_2$) can be constructed, in which $\Theta_1 \cup \Theta_2'$ are valid. Thus the set $\Theta_1 \cup \Theta_2' = \mathsf{Rnm}(\Theta_1 \cup \Theta_2)$ is classically consistent, in contradiction to the coherence of $G$.

**Remark:** The construction of $\mathcal{M}_G$ above is much simpler than the constructions carried out in [2, 23]: a canonical calculus there is first transformed into an equivalent normal form calculus, which is then used to construct the characteristic Nmatrix. The idea is to transform the calculus so that each rule dictates the interpretation for only one $\mathcal{E}$. However, the above definitions show that the transformation into normal form is actually not necessary and we can construct $\mathcal{M}_G$ directly from $G$.

Next we demonstrate the construction of a characteristic 2Nmatrix for some coherent canonical calculi.

**Examples 4.4.** (1) It is easy to see that for any canonical coherent calculus $G$ including the standard (1,1)-ary rules for $\forall$ and $\exists$ from Example 2.5-2:

$$\tilde{\forall}_{\mathcal{M}_G}[\{t,f\}] = \tilde{\forall}_{\mathcal{M}_G}[\{f\}] = \tilde{\exists}_{\mathcal{M}_G}[\{f\}] = \{f\}$$
$$\tilde{\forall}_{\mathcal{M}_G}[\{t\}] = \tilde{\exists}_{\mathcal{M}_G}[\{t,f\}] = \tilde{\exists}_{\mathcal{M}_G}[\{t\}] = \{t\}$$

(2) Consider the canonical calculus $G'$ consisting of the following three $(1,2)$-ary rules from Example 2.5-3:

$$\{p_1(v_1) \Rightarrow p_2(v_1)\}/\Rightarrow \overline{\forall} v_1\ (p_1(v_1), p_2(v_1))$$
$$\{p_2(c) \Rightarrow\ ,\ \Rightarrow p_1(c)\}/\overline{\forall} v_1(p_1(v_1), p_2(v_1)) \Rightarrow$$
$$\{\Rightarrow p_1(c)\ ,\ \Rightarrow p_2(c)\}/\Rightarrow \overline{\exists} v_1(p_1(v_1), p_2(v_1))$$

$G'$ is obviously coherent. The 2Nmatrix $\mathcal{M}_{G'}$ is defined as follows for every $H \in P^+(\{t,f\}^2)$:

$$\tilde{\overline{\forall}}[H] = \begin{cases} \{t\} & \textit{if } \langle t,f\rangle \notin H \\ \{f\} & \textit{otherwise} \end{cases} \qquad \tilde{\overline{\exists}}[H] = \begin{cases} \{t\} & \textit{if } \langle t,t\rangle \in H \\ \{t,f\} & \textit{otherwise} \end{cases}$$

The first rule dictates the condition that $\overline{\forall}[H] = \{t\}$ for the case of $\langle t,f\rangle \notin H$. The second rule dictates the condition that $\overline{\forall}[H] = \{f\}$ for the case that $\langle t,f\rangle \in H$.

Since $G'$ is coherent, these conditions are non-contradictory. The third rule dictates the condition that $\tilde{\exists}[H] = \{t\}$ in the case that $\langle t, t\rangle \in H$. There is no rule which dictates conditions for the case of $\langle t, t\rangle \notin H$, and so the interpretation in this case is non-deterministic.

(3) Consider the canonical calculus $G''$ consisting of the following $(1, 3)$-ary rule:

$$\{p_2(v_1), p_3(v_1) \Rightarrow\}/\mathcal{Q}v_1(p_1(v_1), p_2(v_1), p_3(v_1)) \Rightarrow$$

Of course, $G''$ is coherent. The 2Nmatrix $\mathcal{M}_{G''}$ is defined as follows for every $H \in P^+(\{t, f\}^2)$:

$$\tilde{\tilde{\forall}}[H] = \begin{cases} \{f\} & \textit{if } H \subseteq \{\langle t,t,f\rangle, \langle t,f,t\rangle, \langle t,f,f\rangle, \langle f,t,f\rangle, \langle f,f,t\rangle, \langle f,f,f\rangle\} \\ \{t,f\} & \textit{if } \langle f,t,t\rangle \in H \textit{ or } \langle t,t,t\rangle \in H \end{cases}$$

Now we come to the main theorem, establishing a connection between the coherence of a canonical calculus $G$, the existence of a strongly characteristic 2Nmatrix for $G$ and *strong cut-elimination* in $G$ in the sense of [1].

**Definition 4.5.** Let $G$ be a canonical calculus and let $\mathcal{S}$ be a set of sequents closed under substitution. A proof $P$ of $\Gamma \Rightarrow \Delta$ from $\mathcal{S}$ in $G$ is *simple* if all cuts in $P$ are on formulas from $\mathcal{S}$.

**Definition 4.6.** A calculus $G$ admits *strong cut-elimination*[7] if for every set of sequents $\mathcal{S}$ closed under substitution and every sequent $\Gamma \Rightarrow \Delta$, such that $\mathcal{S} \cup \{\Gamma \Rightarrow \Delta\}$ satisfies the free-variable condition[8]: if $\mathcal{S} \vdash_G \Gamma \Rightarrow \Delta$, then $\Gamma \Rightarrow \Delta$ has a simple proof in $G$.

Note that strong cut-elimination implies standard cut-elimination (which corresponds to the case of an empty set $\mathcal{S}$).

**Theorem 4.7.** *Let $G$ be a canonical calculus. Then the following statements concerning $G$ are equivalent:*

(1) *$G$ is coherent.*
(2) *$G$ has a strongly characteristic 2Nmatrix.*
(3) *$G$ admits strong cut-elimination.*

*Proof.* First we prove that (2) $\Rightarrow$ (1).
Suppose that $G$ has a strongly characteristic 2Nmatrix $\mathcal{M}$. Assume by contradiction that $G$ is not coherent. Then there exist two dual $(n, k)$-ary rules $R_1 = \Theta_1/ \Rightarrow A$ and $R_2 = \Theta_2/A \Rightarrow$ in $G$, such that $\mathsf{Rnm}(\Theta_1 \cup \Theta_2)$ is classically consistent. Suppose that $k = 1$. Then $A = \mathcal{Q}v_1(p_1(v_1), ..., p_n(v_1))$. Recall that $\mathsf{Rnm}(\Theta_1 \cup \Theta_2) = \Theta_1 \cup \Theta_2'$, where $\Theta_2'$ is obtained from $\Theta_2$ by renaming constants and variables that occur also in $\Theta_1$ (see defn. 2.8). For simplicity[9] we assume that the fresh constants used for renaming are all in $L$. Let $\Theta_1 = \{\Sigma_j^1 \Rightarrow \Pi_j^1\}_{1\leq j\leq m}$ and $\Theta_2' = \{\Sigma_j^2 \Rightarrow \Pi_j^2\}_{1\leq j\leq r}$. Since $\Theta_1 \cup \Theta_2'$ is classically consistent, there exists an $L_k^n$-structure $\mathcal{N} = \langle D, I\rangle$, in which both $\Theta_1$ and $\Theta_2'$ are valid. Recall that we also assume that $L_k^n$ is a subset of $L$[10] and so the following are applications of $R_1$ and

[7] [1] does not assume that $\mathcal{S}$ is closed under substitution. Instead, a structural substitution rule is added and the allowed cuts are on substitution instances of formulas from $\mathcal{S}$.

[8] See section 1.

[9] This assumption is not necessary and is used only for simplification of presentation, since we can instantiate the constants by any $L$-terms.

[10] This assumption is again not essential for the proof, but it simplifies the presentation.

$R_2$ respectively:

$$\frac{\{\Sigma^1_j \Rightarrow \Pi^1_j\}_{1\leq j\leq m}}{\Rightarrow \mathcal{Q}v_1(p_1(v_1),...,p_n(v_1))} \qquad \frac{\{\Sigma^2_j \Rightarrow \Pi^2_j\}_{1\leq j\leq m}}{\mathcal{Q}v_1(p_1(v_1),...,p_n(v_1)) \Rightarrow}$$

Let $S$ be any extension of $\mathcal{N}$ to $L$ and $v$ - any $\mathcal{M}$-legal $S$-valuation. It is easy to see that the premises of the applications above are $\mathcal{M}$-valid in $\langle S, v\rangle$ (since the premises contain atomic formulas). Since $G$ is strongly sound for $\mathcal{M}$, both $\Rightarrow \mathcal{Q}v_1(p_1(v_1),...,p_n(v_1))$ and $\mathcal{Q}v_1(p_1(v_1),...,p_n(v_1)) \Rightarrow$ should also be $\mathcal{M}$-valid in $\langle S, v\rangle$, which is of course impossible. The proof for the case of $k = 0$ is simpler and is left to the reader.

Next, we prove that $\underline{(3) \Rightarrow (1)}$.
Let $G$ be a canonical calculus which admits strong cut-elimination. Suppose by contradiction that $G$ is not coherent. Then there are two dual rules of $G$: $\Theta_1/\Rightarrow A$ and $\Theta_2/A \Rightarrow$, such that $\mathsf{Rnm}(\Theta_1 \cup \Theta_2)$ is classically consistent. Let $\Theta$ be the minimal set of clauses, such that $\mathsf{Rnm}(\Theta_1 \cup \Theta_2) \subseteq \Theta$ and $\Theta$ is closed under substitutions. $\Theta \cup \{\Rightarrow\}$ satisfy the free-variable condition, since only atomic formulas are involved and no variables are bound there. It is easy to see that $\Theta \vdash_G \Rightarrow A$ and $\Theta \vdash_G A \Rightarrow$. By using cut, $\Theta \vdash_G \Rightarrow$. But $\Rightarrow$ has no simple proof in $G$ from $\Theta$ (since $\mathsf{Rnm}(\Theta_1 \cup \Theta_2)$ is consistent and $\Theta$ is its closure under substitutions), in contradiction to the fact that $G$ admits strong cut-elimination.

To show both $\underline{(1) \Rightarrow (2)}$ and $\underline{(1) \Rightarrow (3)}$, we will first prove the following proposition.

**Proposition 4.8.** *Let $G$ be a coherent calculus. Let $\mathcal{S}$ be a set of sequents closed under substitution and $\Gamma \Rightarrow \Delta$ - a sequent, such that $\mathcal{S} \cup \{\Gamma \Rightarrow \Delta\}$ satisfies the free-variable condition. If $\Gamma \Rightarrow \Delta$ has no simple proof from $\mathcal{S}$ in $G$, then $\mathcal{S} \not\vdash_{\mathcal{M}} \Gamma \Rightarrow \Delta$.*

*Proof.* Let $\mathcal{S}$ be a set of sequents closed under substitution and $\Gamma \Rightarrow \Delta$ - a sequent, such that $\mathcal{S} \cup \{\Gamma \Rightarrow \Delta\}$ satisfies the free-variable condition. Suppose that $\Gamma \Rightarrow \Delta$ has no simple proof from $\mathcal{S}$ in $G$. To show that $\mathcal{S} \not\vdash_{\mathcal{M}} \Gamma \Rightarrow \Delta$, we will construct a structure $S$ and an $\mathcal{M}$-legal valuation $v$, such that the sequents of $\mathcal{S}$ are $\mathcal{M}$-valid in $\langle S, v\rangle$, while $\Gamma \Rightarrow \Delta$ is not. It is easy to see that we can limit ourselves to the language $L^*$, which is a subset of $L$, consisting of all the constants and predicate and function symbols, occurring in $\mathcal{S} \cup \{\Gamma \Rightarrow \Delta\}$. Let $\mathbf{T}$ be the set of all the terms in $L^*$ which do not contain variables occurring bound in $\Gamma \Rightarrow \Delta$ and $\mathcal{S}$. It is a standard matter to show that $\Gamma, \Delta$ can be extended to two (possibly infinite) sets $\Gamma', \Delta'$ (where $\Gamma \subseteq \Gamma'$ and $\Delta \subseteq \Delta'$), satisfying the following properties:

(1) For every finite $\Gamma_1 \subseteq \Gamma'$ and $\Delta_1 \subseteq \Delta'$, $\Gamma_1 \Rightarrow \Delta_1$ has no simple proof in $G$.
(2) There are no $\psi \in \Gamma'$ and $\varphi \in \Delta'$, such that $\psi \equiv_\alpha \varphi$.
(3) If $\{\Sigma_j \Rightarrow \Pi_j\}_{1\leq j\leq m}/\mathcal{Q}(r)$ is an $(n, 0)$-ary rule of $G$ and $\mathcal{Q}\,(\psi_1, ..., \psi_n) \in ite(r, \Delta', \Gamma')$, then there is some $1 \leq j \leq m$, such that whenever $p_i \in ite(s, \Sigma_j, \Pi_j)$, $\psi_i \in ite(s, \Gamma', \Delta')$ for $s \in \{t, f\}$.
(4) If $\{\Sigma_j \Rightarrow \Pi_j\}_{1\leq j\leq m}/\mathcal{Q}(r)$ is an $(n, 1)$-ary rule of $G$ and $\mathcal{Q}z\,(\psi_1, ..., \psi_n) \in ite(r, \Delta', \Gamma')$, then there is some $1 \leq j \leq m$, such that:
   - For every constant $c$, whenever $p_i(c) \in ite(s, \Sigma_j, \Pi_j)$ for some $1 \leq i \leq n$, then $\psi_i\{\mathbf{t}/z\} \in ite(s, \Gamma', \Delta')$ for every term $\mathbf{t} \in \mathbf{T}$.
   - For every variable $y$, there exists some $\mathbf{t}_y \in \mathbf{T}$, such that whenever $p_i(y) \in ite(s, \Sigma_j, \Pi_j)$ for some $1 \leq i \leq n$, then $\psi_i\{\mathbf{t}_y/z\} \in ite(s, \Gamma', \Delta')$.

Note that every $\mathbf{t} \in \mathbf{T}$ is free for $z$ in $\psi_i$ for every $1 \leq i \leq n$.

(5) For every formula $\psi$ occurring in $\mathcal{S}$, $\psi \in \Gamma' \cup \Delta'$.

Note that the last condition can be satisfied because cuts on formulas from $\mathcal{S}$ are allowed in a simple proof.

Let $S = \langle D, I \rangle$ be the $L^*$-structure defined as follows:

- $D = \mathbf{T}$.
- $I[c] = c$ for every constant $c$ of $L^*$.
- $I[f][\mathbf{t}_1, ..., \mathbf{t}_n] = f(\mathbf{t}_1, ..., \mathbf{t}_n)$ for every $n$-ary function symbol $f$.
- $I[p][\mathbf{t}_1, ..., \mathbf{t}_n] = t$ iff $p(\mathbf{t}_1, ..., \mathbf{t}_n) \in \Gamma'$ for every $n$-ary predicate symbol $p$.

Let $\sigma^*$ be any $S$-substitution satisfying $\sigma^*[x] = \overline{x}$ for every $x \in \mathbf{T}$. (Note that every $x \in \mathbf{T}$ is also a member of the domain and thus has an individual constant referring to it in $L^*(D)$.)

For an $L(D)$-formula $\psi$ (an $L(D)$-term $\mathbf{t}$), we will denote by $\widehat{\psi}$ ($\widehat{\mathbf{t}}$) the $L$-formula ($L$-term) obtained from $\psi$ ($\mathbf{t}$) by replacing every individual constant of the form $\overline{\mathbf{s}}$ for some $\mathbf{s} \in \mathbf{T}$ by the term $\mathbf{s}$. More formally, $\widehat{\mathbf{t}}$ and $\widehat{\psi}$ are defined as follows:

- $\widehat{x} = x$ for any variable $x$ of $L$.
- $\widehat{c} = c$ for any constant $c$ of $L$.
- $\widehat{\overline{\mathbf{t}}} = \mathbf{t}$ for any $\mathbf{t} \in \mathbf{T}$.
- $f(\widehat{\mathbf{t}_1, ..., \mathbf{t}}_n) = f(\widehat{\mathbf{t}}_1, ..., \widehat{\mathbf{t}}_n)$.
- $p(\widehat{\mathbf{t}_1, ..., \mathbf{t}}_n) = p(\widehat{\mathbf{t}}_1, ..., \widehat{\mathbf{t}}_n)$.
- $\mathcal{Q}(\widehat{\psi_1, ..., \psi}_n) = \mathcal{Q}(\widehat{\psi}_1, ..., \widehat{\psi}_n)$.
- $\mathcal{Q}x(\widehat{\psi_1, ..., \psi}_n) = \mathcal{Q}x(\widehat{\psi}_1, ..., \widehat{\psi}_n)$.

**Lemma 4.9.** *Let $\mathbf{t}$ be an $L(D)$-term and $\psi$ - an $L(D)$-formula.*

(1) *For any $z, x$: $\widehat{\mathbf{t}}\{z/x\} = \widehat{\mathbf{t}\{z/x\}}$ and $\widehat{\psi}\{z/x\} = \widehat{\psi\{z/x\}}$.*

(2) *$\psi \sim^S \sigma^*[\widehat{\psi}]$.*

(3) *For every $\psi \in \Gamma' \cup \Delta'$: $\widehat{\sigma^*[\psi]} = \psi$.*

*Proof.* The lemma is proved by a tedious induction on $\mathbf{t}$ and $\psi$.

Define the $S$-valuation $v$ as follows:

- $v[p(\mathbf{t}_1, ..., \mathbf{t}_n)] = I[p][I[\mathbf{t}_1], ..., I[\mathbf{t}_n]]$.
- For every $(n, 0)$-ary quantifier $\mathcal{Q}$ of $L$, if there is some $C \in \Gamma' \cup \Delta'$, such that $C \equiv_\alpha \mathcal{Q}(\widehat{\psi_1, ..., \psi}_n)$, then $v[\mathcal{Q}(\psi_1, ..., \psi_n)] = t$ iff $C \in \Gamma'$. Otherwise $v[\mathcal{Q}(\psi_1, ..., \psi_n)] = t$ iff $\tilde{\mathcal{Q}}[\{\langle v[\psi_1], ..., v[\psi_n] \rangle\}] = \{t\}$.
- For every $(n, 1)$-ary quantifier $\mathcal{Q}$ of $L$, if there is some $C \in \Gamma' \cup \Delta'$, such that $C \equiv_\alpha \mathcal{Q}x(\widehat{\psi_1, ..., \psi}_n)$, then $v[\mathcal{Q}x(\psi_1, ..., \psi_n)] = t$ iff $C \in \Gamma'$. Otherwise $v[\mathcal{Q}x(\psi_1, ..., \psi_n)] = t$ iff $\tilde{\mathcal{Q}}[\{\langle v[\psi_1\{\overline{a}/x\}], ..., v[\psi_n\{\overline{a}/x\}] \rangle \mid a \in D\}] = \{t\}$.

**Lemma 4.10.** (1) *$I^*[\sigma^*[\mathbf{t}]] = \mathbf{t}$ for every $\mathbf{t} \in \mathbf{T}$.*

(2) *For every two $L(D)$-formulas $\psi, \psi'$: if $\psi \equiv_\alpha \psi'$, then $\sigma^*[\psi] \equiv_\alpha \sigma^*[\psi']$.*

(3) *For every two $L(D)$-sentences $\psi, \psi'$: if $\psi \sim^S \psi'$, then $\widehat{\psi} \equiv_\alpha \widehat{\psi'}$.*

*Proof.* The claims are proven by induction on $\mathbf{t}$ in the first case, and on $\psi$ and $\psi'$ in the second and third cases.

**Lemma 4.11.** *For every $\psi \in \Gamma' \cup \Delta'$: $v(\sigma^*[\psi]) = t$ iff $\psi \in \Gamma'$.*

*Proof.*
- If $\psi = p(\mathbf{t}_1, ..., \mathbf{t}_n)$, then $v[\sigma^*[\psi]] = I[p][I[\sigma^*[\mathbf{t}_1]], ..., I[\sigma^*[\mathbf{t}_n]]]$. Note[11] that for every $1 \leq i \leq n$, $\mathbf{t}_i \in \mathbf{T}$. By lemma 4.10-1, $I[\sigma^*[\mathbf{t}_i]] = \mathbf{t}_i$, and by the definition of $I$, $v[\sigma^*[\psi]] = t$ iff $p(\mathbf{t}_1, ..., \mathbf{t}_n) \in \Gamma'$.
- Otherwise $\psi = Q(\psi_1, ..., \psi_n)$ or $\psi = Q'x(\psi_1, ..., \psi_n)$. If $\psi \in \Gamma'$, then by lemma 4.9-3 $\widehat{\sigma^*[\psi]} = \psi \in \Gamma'$ and so $v[\sigma^*[\psi]] = t$. If $\psi \in \Delta'$ then by property 2 of $\Gamma' \cup \Delta'$ it cannot be the case that there is some $C \in \Gamma'$, such that $C \equiv_\alpha \widehat{\sigma^*[\psi]} = \psi$ and so $v[\sigma^*[\psi]] = f$.

**Lemma 4.12.** *$v$ is legal in $\mathcal{M}_G$.*

*Proof.* First we need to show that $v$ respects the $\sim^S$-relation. We prove by induction on $L^*(D)$-sentences $\psi, \psi'$: if $\psi \sim^S \psi'$, then $v[\psi] = v[\psi']$.
- $\psi = p(\mathbf{t}_1, ..., \mathbf{t}_n)$, $\psi' = p(\mathbf{s}_1, ..., \mathbf{s}_n)$ and $\mathbf{t}_i \sim^S \mathbf{s}_i$ for every $1 \leq i \leq n$. Then $I[\mathbf{t}_i] = I[\mathbf{s}_i]$ and by definition of $v$: $v[p(\mathbf{t}_1, ..., \mathbf{t}_n)] = I[p][I[\mathbf{t}_1], ..., I[\mathbf{t}_n]] = I[p][I[\mathbf{s}_1], ..., I[\mathbf{s}_n]] = v[p(\mathbf{s}_1, ..., \mathbf{s}_n)]$.
- $\psi = \mathcal{Q}x(\psi_1, ..., \psi_n)$, $\psi' = \mathcal{Q}y(\psi'_1, ..., \psi'_n)$ and for every $1 \leq i \leq n$: $\psi_i\{z/x\} \sim^S \psi'_i\{z/y\}$ for a fresh variable $z$. Then by lemma 3.6-2 for every $a \in D$: $\psi_i\{z/x\}\{\overline{a}/z\} = \psi_i\{\overline{a}/x\} \sim^S \psi'_i\{\overline{a}/y\} = \psi_i\{z/y\}\{\overline{a}/z\}$. By the induction hypothesis, $\{\langle v[\psi_1\{\overline{a}/x\}], ..., v[\psi_n\{\overline{a}/x\}]\rangle \mid a \in D\} = \{\langle v[\psi'_1\{\overline{a}/x\}], ..., v[\psi'_n\{\overline{a}/x\}]\rangle \mid a \in D\}$. One of the following cases holds:
  - There is no $C \in \Gamma' \cup \Delta'$, such that $C \equiv_\alpha \widehat{\psi}$ or $C \equiv_\alpha \widehat{\psi'}$. Then $v[\mathcal{Q}x(\psi_1, ..., \psi_n)] = t$ iff $\{\langle v[\psi_1\{\overline{a}/x\}], ..., v[\psi_n\{\overline{a}/x\}]\rangle \mid a \in D\} = t$ iff $\{\langle v[\psi'_1\{\overline{a}/x\}], ..., v[\psi'_n\{\overline{a}/x\}]\rangle \mid a \in D\} = t$ iff $v[\mathcal{Q}y(\psi'_1, ..., \psi'_n)] = t$.
  - There is some $C \in \Gamma' \cup \Delta'$, such that $C \equiv_\alpha \widehat{\psi}$. By lemma 4.10-3, $\widehat{\psi} \equiv_\alpha \widehat{\psi'}$, and so $v[\psi] = v[\psi'] = t$ iff $C \in \Gamma$.
  - There is some $C \in \Gamma' \cup \Delta'$, such that $C \equiv_\alpha \widehat{\psi'}$. Similarly to the previous case, $v[\psi] = v[\psi'] = t$ iff $C \in \Gamma$.
- The case of $\psi = \mathcal{Q}(\psi_1, ..., \psi_n)$, $\psi' = \mathcal{Q}(\psi'_1, ..., \psi'_n)$ is treated similarly.

It remains to show that $v$ respects the interpretations of the $(n, k)$-ary quantifiers in $\mathcal{M}_G$. The case of $k = 0$ is not hard and is left to the reader. We will show the proof for the case of $k = 1$. Suppose by contradiction that there is some $L^*(D)$-sentence $A = \mathcal{Q}z(\psi_1, ..., \psi_n)$, such that $v[A] \notin \tilde{Q}[H_A]$, where $H_A = \{\langle v[\psi_1\{\overline{a}/z\}], ..., v[\psi_n\{\overline{a}/z\}]\rangle \mid a \in D\}$. From the definition of $v$, it must be the case that[12]:

(a) there is some $L$-formula $C \in \Gamma' \cup \Delta'$, such that $C \equiv_\alpha \widehat{A}$, and $v[A] = t$ iff $C \in \Gamma'$.

Suppose that $\tilde{Q}[H_A] = \{t\}$ and $v[A] = f$. By definition of $\mathcal{M}_G$ and the fact that $\tilde{Q}[H_A]$ is a singleton, it must be the case that there is some canonical rule $\{\Sigma_k \Rightarrow \Pi_k\}_{1 \leq k \leq m} / \Rightarrow \mathcal{Q}v_1(p_1(v_1), ..., p_n(v_1))$ in $G$, such that:

[11] This is obvious if $\mathbf{t}_i$ does not occur in the set $\{\Gamma \Rightarrow \Delta\} \cup \mathcal{S}$. If it occurs in this set, then by the free-variable condition $\mathbf{t}_i$ does not contain variables bound in this set and so $\mathbf{t}_i \in \mathbf{T}$ by definition of $\mathbf{T}$.

[12] If there is no $L$-formula $C \in \Gamma' \cup \Delta'$, such that $C \equiv_\alpha \widehat{A}$, then by definition of $v$, $v[A]$ is always in $\tilde{\mathcal{Q}}[H_A]$, so this case is not possible.

(b) $\{\Sigma_k \Rightarrow \Pi_k\}_{1\leq k\leq m}$ is valid in a $H_A$-characteristic structure $\mathcal{N} = \langle D_{\mathcal{N}}, I_{\mathcal{N}}\rangle$.

$A = \mathcal{Q}z(\psi_1, ..., \psi_n)$ and $C \equiv_\alpha \widehat{A}$, so $C$ is of the form $\mathcal{Q}w(\varphi_1, ..., \varphi_n)$. By lemma 4.10-2, $\sigma^*[C] \equiv_\alpha \sigma^*[\widehat{A}]$. By lemma 3.6-1, $\sigma^*[C] \sim^S \sigma^*[\widehat{A}]$. By lemma 4.9-2, $\sigma^*[\widehat{A}] \sim^S A$, and thus $\sigma^*[C] \sim^S A$. Let $\phi_i$ be the formula obtained from $\varphi_i$ by substituting every $x \in Fv[\varphi_i] = \{w\}$ for $\sigma^*[x]$. By lemma 3.6-2, $\phi_i\{\overline{a}/w\} \sim^S \psi_i\{\overline{a}/z\}$ for every $a \in D$. We have already shown that $v$ respects the $\sim^S$-relation, and so $v[\phi_i\{\overline{a}/w\}] = v[\psi_i\{\overline{a}/z\}]$. Thus $H_A = \{\langle v[\phi_1\{\overline{a}/w\}], ..., v[\phi_n\{\overline{a}/w\}]\rangle \mid a \in D\}$.
Since $v[A] = f$, it follows from (a) that $C = \mathcal{Q}w(\varphi_1, ..., \varphi_n) \in \Delta'$. Then by property 3 of $\Gamma' \cup \Delta'$, there is some $1 \leq j \leq m$, such that whenever $p_i(y) \in ite(r, \Sigma_j, \Pi_j)$, there is some $\mathbf{t}_y \in \mathbf{T}$, such that $\varphi_i\{\mathbf{t}_y/w\} \in ite(r, \Gamma', \Delta')$. By lemma 4.11, $v[\sigma^*[\varphi_i\{\mathbf{t}_y/w\}]] = v[\phi_i\{\sigma^*[\mathbf{t}_y]/w\}] = r$. Since $\mathcal{N}$ is $H_A$-characteristic, there is some $a_y \in D_{\mathcal{N}}$, such that $I_{\mathcal{N}}[p_i][a_y] = v[\phi_i\{\sigma^*[\mathbf{t}_y]/w\}] = r$.
Let us now show that $\Sigma_j \Rightarrow \Pi_j$ is not valid in $\mathcal{N}$ (in contradiction to (b)). Let $\mu$ be any $\mathcal{N}$-substitution, such that $\mu[y] = \overline{a}_y$ for every variable $y$ occurring in $\Sigma_j \cup \Pi_j$. We now show that whenever $p(\mathbf{t}) \in ite(s, \Sigma_j, \Pi_j)$, $I[p][I[\mu[\mathbf{t}]]] = s$.
Let $p(\mathbf{t}) \in ite(s, \Sigma_j, \Pi_j)$. If $\mathbf{t}$ is some variable $y$, then $I_{\mathcal{N}}[p_i][\mu[y]] = I_{\mathcal{N}}[p_i][I_{\mathcal{N}}[\overline{a}_y]] = I_{\mathcal{N}}[p_i][a_y] = s$. Otherwise $\mathbf{t}$ is some constant $c$. By property 3 of $\Gamma' \cup \Delta'$, for every $\mathbf{t} \in \mathbf{T}$: $\varphi_i\{\mathbf{t}/x\} \in ite(s, \Sigma_j, \Pi_j)$. By lemma 4.11, $v[\sigma^*[\varphi_i\{\mathbf{t}/w\}]] = v[\phi_i\{\sigma^*[\mathbf{t}]/w\}] = s$. Thus for every $\mathbf{t} \in \mathbf{T}$: $v[\phi_i\{\sigma^*[\mathbf{t}]/w\}] = v[\phi_i\{\overline{\mathbf{t}}/w\}] = s$. Since $\mathcal{N}$ is $H_A$-characteristic, $I_{\mathcal{N}}[p_c][I_{\mathcal{N}}[c]] = s$. And so we have shown that $\Sigma_j \Rightarrow \Pi_j$ is not valid in $\mathcal{N}$, in contradiction to (b).
The proof for the case of $\tilde{Q}[H_A] = \{f\}$ and $v[A] = t$ is symmetric.

**Lemma 4.13.** *For every sequent $\Sigma \Rightarrow \Pi \in \mathcal{S}$, $\Sigma \Rightarrow \Pi$ is $\mathcal{M}$-valid in $\langle S, v\rangle$.*

*Proof.* Suppose by contradiction that there is some $\Sigma \Rightarrow \Pi \in \mathcal{S}$, which is not $\mathcal{M}$-valid in $\langle S, v\rangle$. Then there exists some $S$-substitution $\mu$, such that for every $\psi \in \Sigma$: $S, v \models_{\mathcal{M}} \mu[\psi]$, and for every $\varphi \in \Pi$: $S, v \not\models_{\mathcal{M}} \mu[\varphi]$. Note that for every $\phi \in \Sigma \cup \Pi$, $\widehat{\mu[\phi]}$ is a substitution instance of $\phi$. Since $\mathcal{S}$ is closed under substitution, $\widehat{\mu[\phi]}$ also occurs in $\mathcal{S}$, and thus by property 5 of $\Gamma' \cup \Delta'$: $\widehat{\mu[\phi]} \in \Gamma' \cup \Delta'$. By lemma 4.11, if $\widehat{\mu[\phi]} \in \Gamma'$ then $v[\sigma^*[\widehat{\mu[\phi]}]] = t$, and if $\widehat{\mu[\phi]} \in \Delta'$ then $v[\sigma^*[\widehat{\mu[\phi]}]] = f$. By lemma 4.9-2, $\mu[\phi] \sim^S \sigma^*[\widehat{\mu[\phi]}]$. Since $v$ is $\mathcal{M}$-legal, it respects the $\sim^S$-relation and so for every $\phi \in \Sigma \cup \Pi$: $v[\mu[\phi]] = v[\sigma^*[\widehat{\mu[\phi]}]]$. Thus $\widehat{\mu[\Sigma]} \subseteq \Gamma'$ and $\widehat{\mu[\Pi]} \subseteq \Delta'$. But $\widehat{\mu[\Sigma]} \Rightarrow \widehat{\mu[\Pi]}$ has a simple proof from $\mathcal{S}$ in $G$, in contradiction to property 1 of $\Gamma' \cup \Delta'$.

We have shown that (i) $v$ is legal in $\mathcal{M}$, (ii) for every $\psi \in \Gamma' \cup \Delta'$: $v[\sigma^*[\psi]] = t$ iff $\psi \in \Gamma'$, and (iii) the sequents in $\mathcal{S}$ are $\mathcal{M}$-valid in $\langle S, v\rangle$. From (ii) it follows that $\Gamma \Rightarrow \Delta$ is not $\mathcal{M}$-valid in $\langle S, v\rangle$, which completes the proof. □

Now we prove (1) $\Rightarrow$ (2):
Suppose that $G$ is coherent. Let us show that $\mathcal{M}_G$ is a strongly characteristic 2Nmatrix for $G$. By definition of $\mathcal{M}_G$, it is suitable for $G$ (see defn. 4.1). By theorem 4.2, $G$ is strongly sound for $\mathcal{M}_G$.

For strong completeness, let $\mathcal{S}$ be a set of sequents closed under substitution. Suppose that a sequent $\Gamma \Rightarrow \Delta$ has no proof from $\mathcal{S}$ in $G$. If $\mathcal{S} \cup \{\Gamma \Rightarrow \Delta\}$ does not satisfy the free-variable condition, obtain $\mathcal{S}' \cup \{\Gamma' \Rightarrow \Delta'\}$ by renaming the bound variables, so that $\mathcal{S}' \cup \{\Gamma' \Rightarrow \Delta'\}$ satisfies the condition (otherwise, take $\Gamma' \Rightarrow \Delta'$ and $\mathcal{S}'$ to be $\Gamma \Rightarrow \Delta$ and $\mathcal{S}$ respectively). Then $\Gamma' \Rightarrow \Delta'$ has no proof from $\mathcal{S}'$ in $G$ (otherwise we could obtain a proof of $\Gamma \Rightarrow \Delta$ from $\mathcal{S}$ by using cuts on logical axioms), and so it also has no simple proof from $\mathcal{S}'$ in $G$. By proposition 4.8, $\mathcal{S}' \not\vdash_{\mathcal{M}} \Gamma' \Rightarrow \Delta'$. That is, there is an $L$-structure $S$ and an $\mathcal{M}$-legal valuation $v$, such that the sequents in $\mathcal{S}'$ are $\mathcal{M}$-valid in $\langle S, v\rangle$, while $\Gamma' \Rightarrow \Delta'$ is not. Since $v$ respects the $\equiv_\alpha$-relation, the sequents of $\mathcal{S}$ are also $\mathcal{M}$-valid in $\langle S, v\rangle$, while $\Gamma \Rightarrow \Delta$ is not. And so $\mathcal{S} \not\vdash_{\mathcal{M}} \Gamma \Rightarrow \Delta$. We have shown that $G$ is strongly complete (and strongly sound) for $\mathcal{M}_G$. Thus $\mathcal{M}_G$ is a strongly characteristic 2Nmatrix for $G$.

Finally, we prove that $\underline{(1) \Rightarrow (3)}$.
Let $G$ be a coherent calculus. Let $\mathcal{S}$ be a set of sequents closed under substitution, and let $\Gamma \Rightarrow \Delta$ be a sequent, such that $\mathcal{S} \cup \{\Gamma \Rightarrow \Delta\}$ satisfies the free-variable condition. Suppose that $\mathcal{S} \vdash_G \Gamma \Rightarrow \Delta$. We have already shown above that $\mathcal{M}_G$ is a strongly characteristic 2Nmatrix for $G$. Thus $\mathcal{S} \vdash_{\mathcal{M}} \Gamma \Rightarrow \Delta$, and by proposition 4.8, $\Gamma \Rightarrow \Delta$ has a simple proof from $\mathcal{S}$ in $G$. Thus $G$ admits strong cut-elimination. □

**Remark:** At this point it should be noted that the renaming of clashing constants in the definition of coherence (see defn. 2.9) is crucial. Consider, for instance, a canonical calculus $G$ consisting of the introduction rules $\{p_1(c) \Rightarrow\ ;\ \Rightarrow p_1(c')\}/ \Rightarrow \mathcal{Q}v_1\ p_1(v_1)$ and $\{p_1(c'') \Rightarrow\ ;\ \Rightarrow p_1(c)\}/\mathcal{Q}v_1\ p(v_1) \Rightarrow$ for a (1,1)-ary quantifier $\mathcal{Q}$. Without renaming of clashing constants, we would conclude that the set $\{p_1(c) \Rightarrow\ ;\ \Rightarrow p_1(c')\ ;\ p_1(c'') \Rightarrow, \Rightarrow p_1(c)\}$ is classically inconsistent. However, $G$ obviously has no strongly characteristic 2Nmatrix, since the rules dictate contradicting requirements for $\tilde{\mathcal{Q}}[\{t, f\}]$. But if we perform renaming first, obtaining the set $\mathsf{Rnm}(\Theta_1 \cup \Theta_2) = \{p_1(c) \Rightarrow\ ,\ \Rightarrow p_1(c')\ ,\ p_1(c'') \Rightarrow, \Rightarrow p_1(c''')\}$, we shall see that $\mathsf{Rnm}(\Theta_1 \cup \Theta_2)$ is classically consistent and so $G$ is not coherent. Hence, by the above theorem, $G$ has no strongly characteristic 2Nmatrix.

**Corollary 4.14.** *The existence of a strongly characteristic 2Nmatrix for a canonical calculus $G$ is decidable.*

*Proof.* By theorem 4.7, the question whether $G$ has a strongly characteristic 2Nmatrix is equivalent to the question whether $G$ is coherent, and this, by proposition 2.10, is decidable.

**Remark:** The above results are related to the results in [8], where a general class of sequent calculi with $(n, k)$-ary quantifiers and a (not necessarily standard) set of structural rules called *standard* calculi are defined. A canonical calculus is a particular instance of a standard calculus which includes all of the standard structural rules. [8] formulate syntactic necessary and sufficient conditions for a slightly generalized version of cut-elimination with non-logical axioms. Unlike in this paper, the non-logical axioms must consist of *atomic* formulas (and must be closed under cuts and substitutions). But the results of [8] apply to a much wider class of calculi (since different combinations of structural rules are allowed). In addition, a constructive modular cut-elimination procedure is provided. The reductivity condition of [8] can be shown to be equivalent to our coherence criterion in the context of canonical systems[13] .

4.2. **Coherence and standard cut-elimination.** In the previous subsection we have studied the connection between coherence and strong cut-elimination. In this subsection we focus on standard cut-elimination in canonical calculi. It easily follows from theorem 4.7 that coherence implies cut-elimination:

**Corollary 4.15.** *Let $G$ be a canonical calculus. If $G$ is coherent, then for every sequent $\Gamma \Rightarrow \Delta$ satisfying the free-variable condition: if $\Gamma \Rightarrow \Delta$ is provable in $G$, then it has a cut-free proof in $G$.*

Thus coherence is a sufficient condition for cut-elimination in a canonical calculus. In the more restricted canonical systems of [2, 23] it also is a necessary condition. However, things get more complicated with the more general canonical rules studied in this paper.

**Example 4.16.** Consider, for instance, the following canonical calculus $G_0$ consisting of the following two inference rules: $\Theta_1/ \Rightarrow \mathcal{Q}v_1(p_1(v_1), p_2(v_1))$ and $\Theta_2/\mathcal{Q}v_1(p_1(v_1), p_2(v_1)) \Rightarrow$, where:

$\Theta_1 = \Theta_2 = \{p_1(v_1) \Rightarrow p_2(v_1)\,; \Rightarrow p_1(c_1)\,; \Rightarrow p_2(c_1)\,; p_1(c_2) \Rightarrow; p_2(c_2) \Rightarrow; p_1(c_3) \Rightarrow; \Rightarrow p_2(c_3)\}$

Clearly, $G_0$ is not coherent. We now sketch a proof that the only sequents provable in $G_0$ are logical axioms. This immediately implies that $G_0$ admits cut-elimination.
To prove this it suffices to show that for every rule of $G_0$: if its premises are logical axioms, then its conclusion is a logical axiom. Suppose by contradiction that we can apply e.g. the first rule on logical axioms and obtain a conclusion which is not a logical axiom. Then the application would be of the form:

$$\frac{\Gamma, \chi[p_1]\{\chi[v_1]/w\} \Rightarrow \Delta, \chi[p_2]\{\chi[v_1]/w\} \quad \ldots \quad \Gamma \Rightarrow \chi[p_1]\{\chi[c_1]/w\}, \Delta \quad \Gamma \Rightarrow \chi[p_2]\{\chi[c_1]/w\}, \Delta}{\Gamma \Rightarrow \mathcal{Q}w(\chi[p_1], \chi[p_2]), \Delta}$$

Since the proved sequent is not a logical axiom, (*) there are no $A \in \Gamma$ and $B \in \Delta$, such that $A \equiv_\alpha B$. Moreover, since $\Gamma, \chi[p_1]\{\chi[v_1]/w\} \Rightarrow \Delta, \chi[p_2]\{\chi[y]/w\}$ is a logical axiom, either (i) there is some $C \in \Delta$, such that $C \equiv_\alpha \chi[p_1]\{\chi[v_1]/w\}$, (ii) there is some $C \in \Gamma$, such that $C \equiv_\alpha \chi[p_2]\{\chi[v_1]/w\}$, or (iii) $\chi[p_1](\chi[v_1]/w) \equiv_\alpha \chi[p_2]\{\chi[v_1]/w\}$. Suppose (i) holds, i.e. there is some some $C \in \Delta$, such that $C \equiv_\alpha \chi[p_1]\{\chi[v_1]/w\}$. Then since $\chi[v_1]$ cannot occur free in $\Delta$, $w \notin Fv[C]$, and so $w \notin Fv[\chi[p_1]]$. Hence, $\chi[p_1]\{\chi[c_1]/w\} = \chi[p_1]\{\chi[v_1]/w\} = \chi[p_1]$. Now since $\Gamma \Rightarrow \chi[p_1]\{\chi[c_1]/w\}, \Delta$ is a logical axiom, and due to (*), there is some $D \in \Gamma$, such that $D \equiv_\alpha \chi[p_1]\{\chi[c_1]/w\}$. But since $\chi[p_1]\{\chi[c_1]/w\} = \chi[p_1]\{\chi[v_1]/w\}$, $C \equiv_\alpha D$, $C \in \Delta$ and $D \in \Gamma$, in contradiction to (*). The case (ii) is treated similarly using the constant $c_2$. The case (iii) is handled using the constant $c_3$.
Thus, only logical axioms are provable in $G_0$ and so it admits standard cut-elimination, although it is not coherent.

Hence coherence is not a necessary condition for cut-elimination in general. However, below we characterize a more restricted subclass of canonical systems, for which this property does hold.

[13] We wish to thank Agata Ciabattoni for pointing out these facts to us in a personal correspondence.

**Definition 4.17.** A canonical calculus $G$ is *simple* if for every two dual $(n, k)$-ary canonical rules $\Theta_1/\Rightarrow A$ and $\Theta_2/A \Rightarrow$ one of the following properties holds:

(1) $k = 0$, i.e. $\Theta_1/\Rightarrow A$ and $\Theta_2/A \Rightarrow$ are propositional rules.

(2) $k = 1$ and one of the following holds for each variable $y$ occurring in $\mathsf{Rnm}(\Theta_1 \cup \Theta_2)$:
- There is at most one $1 \leq i \leq n$, such that $y$ occurs in $p_i(y)$ in $\mathsf{Rnm}(\Theta_1 \cup \Theta_2)$ and there is at most one constant $c$, such that $p_i(c)$ also occurs in $\mathsf{Rnm}(\Theta_1 \cup \Theta_2)$.
- There are two different $1 \leq i, j \leq n$, such that $y$ occurs in $p_i(y)$ and $p_j(y)$ in $\mathsf{Rnm}(\Theta_1 \cup \Theta_2)$ and for every constant $c$, there is no such $1 \leq k \leq n$, that both $p_k(y)$ and $p_k(c)$ occur in $\mathsf{Rnm}(\Theta_1 \cup \Theta_2)$.

**Examples 4.18.** (1) All the canonical calculi from examples 2.5 are simple.

(2) Consider the canonical calculus $G_1$, consisting of the following two rules for a $(3, 1)$-ary quantifier $\mathcal{Q}_1$:

$$\{p_1(v_1) \Rightarrow\ ;\ p_1(c), p_2(c) \Rightarrow\ \}/ \Rightarrow \mathcal{Q}_1 v_1(p_1(v_1), p_2(v_1), p_3(v_1))$$

$$\{\Rightarrow p_1(v_1)\ ;\ \Rightarrow p_2(e)\}/\mathcal{Q}_1 v_1(p_1(v_1), p_2(v_1), p_3(v_1)) \Rightarrow$$

It is easy to see that $G_1$ is a simple coherent calculus.

(3) If we modify the first rule of $G_1$ as follows:

$$\{p_1(v_1) \Rightarrow\ ;\ p_1(c), p_2(c) \Rightarrow\ ;\ p_1(d) \Rightarrow p_3(d)\}/ \Rightarrow \mathcal{Q}_1 v_1(p_1(v_1), p_2(v_1), p_3(v_1))$$

the resulting calculus is not simple, since both $p_1(c)$ and $p_1(d)$ occur in the premises of the rule, together with $p_1(v_1)$.

(4) The calculus $G_0$ from example 4.16 is not simple, since for instance $p_1(v_1)$, $p_1(c_1)$ and $p_1(c_5)$ occur in the premises (after renaming).

For a set of clauses $\Theta$, denote by $\Theta\{c/x\}$ the set $\{\Gamma\{c/x\} \Rightarrow \Delta\{c/x\} \mid \Gamma \Rightarrow \Delta \in \Theta\}$. Then the following lemma can be easily proved:

**Lemma 4.19.** *Let $\Theta$ be a classically consistent set of clauses. Then for any constant $c$, $\Theta\{c/x\}$ is also classically consistent.*

**Proposition 4.20.** *If a simple canonical calculus $G$ admits cut-elimination, then it is coherent.*

*Proof.* Suppose that a simple canonical calculus $G$ is not coherent. Then there is a pair of $(n, k)$-ary dual rules $R_1 = \Theta_1/\Rightarrow A$ and $R_2 = \Theta_2/A \Rightarrow$, such that $\mathsf{Rnm}(\Theta_1 \cup \Theta_2)$ is classically consistent. If $k = 0$, then the proof is similar to the proof of theorem 4.7 in [2]. Otherwise, $k = 1$, $A = \mathcal{Q}v_1(p_1(v_1), ..., p_n(v_1))$ and whenever $p_i(y)$ occurs in $\mathsf{Rnm}(\Theta_1 \cup \Theta_2)$ for some variable $y$ and some $1 \leq i \leq n$, there is at most one constant $c$, such that $p_i(c)$ also occurs in $\mathsf{Rnm}(\Theta_1 \cup \Theta_2)$. Recall that $\mathsf{Rnm}(\Theta_1 \cup \Theta_2) = \Theta_1 \cup \Theta'_2$, where $\Theta'_2$ is obtained from $\Theta_2$ by renaming of constants and variables which occur in $\Theta_1$ (see defn. 2.8). We assume that the new constants in $\Theta'_2$ are in $L$ (this assumption is not necessary but it simplifies the presentation).
Obtain the sets $\Upsilon_1, \Upsilon_2$ from $\Theta_1, \Theta'_2$ respectively as follows. For every $1 \leq i \leq n$, if $p_i(c)$ occurs in $\Theta_1 \cup \Theta'_2$ for some constant $c$, replace all variables $y$, such that $p_i(y)$ occurs in $\Theta_1 \cup \Theta'_2$ by $c$ (note that this is well-defined due to the special property of simple calculi). Otherwise, replace all variables $y$, such that $p_i(y)$ occurs in $\Theta_1 \cup \Theta'_2$ by a fresh constant $d_i$ of $L$. Then $\Upsilon = \Upsilon_1 \cup \Upsilon_2$ is obtained from $\Theta_1 \cup \Theta'_2$ by replacing all variables by constants. Since $\Theta_1 \cup \Theta'_2$ is classically consistent, by repeated application of lemma 4.19, $\Upsilon$ is also classically consistent. Then there exists some $L$-structure $S$ in which the set of clauses $\Upsilon$ is (classically)

valid. Since $\Upsilon$ consists of closed atomic formulas, there also exists a (classical) propositional valuation $v_S$, which satisfies $\Upsilon$. Let $\Phi = \{A \mid v_S[A] = t, A \in \Gamma \cup \Delta, \Gamma \Rightarrow \Delta \in \Upsilon\}$ and $\Psi = \{A \mid v_S[A] = f, A \in \Gamma \cup \Delta, \Gamma \Rightarrow \Delta \in \Upsilon\}$. Let $B_j = \{\Pi, \Phi \Rightarrow \Sigma, \Psi \mid \Pi \Rightarrow \Sigma \in \Upsilon_j\}$ for $j = 1, 2$. Then $B_1$ and $B_2$ are sets of standard axioms. (Since $v_S$ satisfies $\Pi \Rightarrow \Sigma$, there is some $A \in \Pi$, such that $v_S[A] = f$, or some $A \in \Delta$, such that $v_S[A] = t$. In the former case, $A \in \Psi$ and in the latter case, $A \in \Phi$.)
Let $x$ be a fresh variable of $L$. Define the $\langle R_1, \Psi \cup \Phi, x\rangle$-mapping $\chi$ (see defn. 2.3) as follows. For every $1 \leq i \leq n$, $\chi[p_i] = p_i(x)$ if there is some constant $c$, such that $p_i(c)$ occurs in $\Theta_1 \cup \Theta'_2$. Otherwise, $\chi[p_i] = p_i(d_i)$ (where $d_i$ is the fresh constant of $L$ chosen above). For every constant $c$ and variable $y$ occurring in $\Theta_1 \cup \Theta'_2$: $\chi[c] = c$ and $\chi[y] = y$. It is easy to see that $\Upsilon_1 = \{\chi[\Sigma'] \Rightarrow \chi[\Pi'] \mid \Sigma' \Rightarrow \Pi' \in \Theta_1\}$ and $\Upsilon_2 = \{\chi[\Sigma'] \Rightarrow \chi[\Pi'] \mid \Sigma' \Rightarrow \Pi' \in \Theta'_2\}$. Thus the following is an application of $R_1$:

$$\frac{B_1}{\Phi, \mathcal{Q}x\ (\chi[p_1], ..., \chi[p_n]) \Rightarrow \Psi}$$

It is easy to check that $\chi$ is also an $\langle R_2, \Psi \cup \Phi, x\rangle$-mapping and so the following is also an application of $R_2$:

$$\frac{B_2}{\Phi \Rightarrow \Psi, \mathcal{Q}x\ (\chi[p_1], ..., \chi[p_n])}$$

By cut, $\Phi \Rightarrow \Psi$ is provable, but $\Phi$ and $\Psi$ are disjoint sets of atomic formulas, thus they have no cut-free proof in $G$, in contradiction to our assumption. □

## 5. Summary and further research

In this paper we have considerably extended the characterization of canonical calculi of [2, 23] to $(n, k)$-ary quantifiers. Focusing on the case of $k \in \{0, 1\}$, we have shown that the following statements concerning a canonical calculus $G$ are equivalent: (i) $G$ is coherent, (ii) $G$ has a strongly characteristic 2Nmatrix, and (iii) $G$ admits strong cut-elimination. We have also shown that coherence is not a necessary condition for standard cut-elimination, and characterized a subclass of canonical systems called *simple* calculi, for which this property does hold.

In addition to these proof-theoretical results for a natural type of multiple conclusion Gentzen-type systems with $(n, 1)$-ary and $(n, 0)$-ary quantifiers, this work also provides further evidence for the thesis that the meaning of a logical constant is given by its introduction (and "elimination") rules . We have shown that at least in the framework of multiple-conclusion consequence relations, any "reasonable" set of canonical quantificational rules completely determines the semantics of the quantifier.

This paper also demonstrates the important role of the semantic framework of Nmatrices ([2, 22]), which substantially contributes to the understanding of the connection between syntactic rules and semantic interpretations of quantifiers. Due to the modularity of the framework, we were able to detect the semantic effect of each of the canonical rules, which of course is not possible using deterministic matrices.

Some of the most immediate research directions are as follows. In the case of $k \in \{0, 1\}$, we still need to characterize the most general subclass of canonical calculi, for which coherence is both a necessary and sufficient condition for standard cut-elimination (it is not clear whether the characterization of simple calculi can be further extended).

Extending these results to the case of $k > 1$ might lead to new insights on Henkin quantifiers and other important generalized quantifiers. However, even for the simplest case of $(1, 2)$-ary quantifiers the extension is far from straightforward.
Consider, for instance, the calculus $G$, consisting of the following two (1,2)-ary rules:

$$\{p(c,x) \Rightarrow\}/ \Rightarrow \mathcal{Q}z_1z_2p(z_1,z_2) \quad \{\Rightarrow p(y,d)\}/\mathcal{Q}z_1z_2p(z_1,z_2) \Rightarrow$$

$G$ is coherent, but it is easy to see that $\mathcal{M}_G$ is not well-defined in this case. And even if a 2Nmatrix $\mathcal{M}$ suitable for $G$ does exist, it is not necessarily sound for $G$. It is clear that the distributional interpretation of quantifiers is no longer adequate for the case of $k > 1$, since it cannot capture any kind of dependencies between elements of the domain. Thus a more general interpretation of quantifiers is needed.

Another important research direction is extending canonical systems with equality. This will allow us to treat counting $(n, k)$-ary quantifiers, like "there are at most two elements $a, b$, such that $p(a, b)$ holds". Clearly, equality must be incorporated also into the representation language $L^n_k$. Standard and strong cut-elimination and its connection to the coherence of canonical systems are yet to be investigated for canonical systems with equality.

## Acknowledgement

This research was supported by the *Israel Science Foundation* founded by the Israel Academy of Sciences and Humanities (grant No 809/06).

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