

On the power of quantum, one round, two prover interactive proof systems

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Abstract

We analyze quantum two prover one round interactive proof systems, in which noninteracting provers can share unlimited entanglement. The maximum acceptance probability is characterized as a superoperator norm. We get some partial results about the superoperator norm, and in particular we analyze the "rank one" case.

1 Introduction

Classical interactive proof systems allow an interaction between an efficient verifier and an all powerful prover. Classical interactive proof systems are quite powerful: one prover can prove theorems in PSPACE to an efficient verifier [12] while two or more powerful provers that cannot interact between themselves can prove the whole of NEXP [2].

Kitaev and Watrous [9] studied the power of interaction between an efficient quantum verifier and a single prover. They prove that such a proof system is at least as powerful as a classical one prover proof system but not as powerful as classical two provers ($\text{PSPACE} \subseteq \text{QIP} \subseteq \text{EXP}$). Moreover they show that in the quantum case 3 communication messages are enough ($\text{QIP} = \text{QIP}(3)$). They also show how to achieve perfect completeness and parallel amplification for the model.

The quantum multiprover case is more complicated. As in the classical case the provers cannot interact between themselves. There are three models concerning the initial state of the provers private qubits. In one model they are not allowed to share any prior entanglement at all, in the second they are allowed to share limited entanglement and in the third they can share unlimited entanglement. Kobayashi and Matsumoto [10] prove that without entanglement quantum multiprover proofs are as powerful as classical. They also prove that

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if we limit prior entanglement to be polynomial in the input size the power of the proof can only decrease.

In this paper we concentrate on the case of two quantum provers with unlimited prior entanglement and one round of communication. The power of such proofs is not known. On the one hand more entanglement gives the provers power to prove more languages to the verifier, but on the other hand it gives them more power to cheat the verifier. The only prior result we are aware of, is that of Kempe and Vidick [7], that such provers can prove NP with perfect completeness and some non-negligible soundness, to a verifier whose space is limited to be logarithmic in the input size.

The problem we are facing touches the basic question of what entanglement can achieve, and how to quantify it. There are many demonstrations of the power of entanglement (e.g., teleportation [3] and superdense coding [4]). There is also a natural measure for measuring the amount of entanglement in *pure states* [11]. Yet, there is no good measure for the amount of entanglement in *mixed states*.

Another demonstration of the power of entanglement are nonlocal games. In those games Alice and Bob play as provers against a fixed verifier. Their goal is to make him accept. The value of the game is the probability a verifier accepts when Alice and Bob play optimally. Alice and Bob cannot interact during the game but in the quantum model they may share prior entanglement. The CHSH and the Magic Square games are two examples presented in [5] and [1] for games in which quantum provers outperform the classical provers and violate Bell inequalities for classical correlation between noninteracting parties. In the case of the Magic Square game there is even a perfect quantum strategy that achieves game value 1. The problem we work on is a strong generalization of quantum nonlocal games.

It is fair to say that entanglement is far from being understood. In particular, we don't even understand whether infinite entanglement gives additional power over limited entanglement, and this is the core of the problem we try to deal with in this work.

Our approach is to generalize the direction Watrous and Kitaev [9] took with the quantum *single* prover case. They gave an algebraic characterization for the maximum acceptance probability of a fixed verifier in terms of the diamond superoperator norm. Then they used a nice algebraic property of the diamond norm, proved previously by Kitaev [8], to get strong results about quantum single prover proofs.

We manage to get an algebraic characterization of one-round, two-prover games. We define a "product superoperator norm" and use it to characterize the maximum acceptance probability of a fixed verifier in the quantum two prover, one round case. However, we are unable to analyze it algebraically. We get some partial results and in particular we analyze the "rank one" case. Even this case is nontrivial. We also present some hypotheses about our characterization and give their implications on the power of the proof system.

2 Preliminaries and Background

2.1 Basic Notation

For a Hilbert space \mathcal{H} with dimension $\dim(\mathcal{H})$ we denote by $L(\mathcal{H})$ the set of all linear operators over \mathcal{H} and by $U(\mathcal{H})$ the set of all unitary operators over \mathcal{H} . $I_{\mathcal{H}}$ denotes the identity operator over \mathcal{H} . A *superoperator* $T : L(\mathcal{H}_1) \rightarrow L(\mathcal{H}_2)$ is a linear mapping from $L(\mathcal{H}_1)$ to $L(\mathcal{H}_2)$.

Definition 1. *The trace out operator is a superoperator from $L(\mathcal{H}_1 \otimes \mathcal{H}_2)$ to $L(\mathcal{H}_1)$ defined by*

$$\text{Tr}_{\mathcal{H}_2}(A \otimes B) = \text{Tr}(B) \cdot A$$

for $A \in L(\mathcal{H}_1)$ and $B \in L(\mathcal{H}_2)$ and extended linearly to all of $L(\mathcal{H}_1 \otimes \mathcal{H}_2)$.

It can be checked that for $X \in L(\mathcal{H}_1 \otimes \mathcal{H}_2)$, $\text{Tr}_{\mathcal{H}_2}(X)$ is independent of the representation $X = \sum_i A_i \otimes B_i$. Also it is easy to check that

$$\text{Tr}(\text{Tr}_{\mathcal{H}_2}(X)) = \text{Tr}(X) \tag{1}$$

and that

$$\text{Tr}_{\mathcal{H}_2}((C \otimes I)X) = C \text{Tr}_{\mathcal{H}_2}(X) \tag{2}$$

for any $C \in L(\mathcal{H}_1)$.

2.2 Quantum Interactive Proof Systems

In quantum interactive proof systems the verifier and the provers are quantum players. The protocol lives in $\mathcal{V} \otimes \mathcal{M}_1 \otimes \dots \otimes \mathcal{M}_k \otimes \mathcal{P}_1 \otimes \dots \otimes \mathcal{P}_k$ where \mathcal{V} is the verifier private register, \mathcal{M}_i is the message register between the verifier and the i 'th prover and \mathcal{P}_i is the i 'th prover private register. \mathcal{V} and \mathcal{M}_i are of size polynomial in the input length. In every round of the proof the verifier applies a unitary transformation on $\mathcal{V} \otimes \mathcal{M}_1 \otimes \dots \otimes \mathcal{M}_k$ after which the \mathcal{M}_i register is sent to the i 'th prover who applies a unitary transformation on $\mathcal{M}_i \otimes \mathcal{P}_i$ and sends \mathcal{M}_i back to the verifier. Because of the safe storage and the locality principle it is convenient to assume without loss of generality that there is only one measurement done by the verifier at the end, based on which he accepts or rejects.

QIP(m) (Quantum IP) is the class of languages that can be proved to a quantum verifier with $c = \frac{2}{3}$ and $s = \frac{1}{3}$ by a single quantum prover with at most m messages passed between the prover and the verifier. Note that in the quantum model we usually count the actual number of passed messages in each direction and not the number of rounds, as is customary in the classical model.

Kitaev and Watrous [9] proved that $\text{PSPACE} \subseteq \text{QIP} = \text{QIP}(3) \subseteq \text{EXP}$. There is no similar result in classical IP. They also showed that any language in QIP(3) has a proof with perfect completeness. Also QIP(3) has perfect parallel amplification.

We now turn to *multiprover proof systems*. An important parameter of multiprover quantum interactive proof systems is the maximal amount of entangled qubits the provers are allowed to share (if at all) in the initial state of $\mathcal{P}_1 \otimes \dots \otimes \mathcal{P}_k$. We say that the provers have $q(|x|)$ -*prior-entanglement* if all the provers hold at most $q(|x|)$ entangled qubits in the initial state.

Definition 2.1. Fix functions $k(|x|), m(|x|), q(|x|) \geq 0$. $QMIP(k, m, q)$ is the class of languages L for which there is an interactive proof system with

- k quantum provers.
- m communication rounds.
- The initial state $|\psi\rangle$, between the provers is $q(|x|)$ -prior-entangled.

such that

1. If $x \in L$ then there exist quantum provers P_1, \dots, P_k and $|\psi\rangle$ for which V_x accepts with probability at least $\frac{2}{3}$.
2. If $x \notin L$ then for all quantum provers P_1, \dots, P_k and $|\psi\rangle$, V_x accepts with probability at most $\frac{1}{3}$.

Note that we define m as the number of communication rounds, and not as the number of communication messages. Since we study only the case of one round two messages, the classical convention is more appropriate in this case.

Denote

$$\begin{aligned} QMIP(k, m) &= QMIP(k, m, 0) \\ QMIP^{poly}(k, m) &= QMIP(k, m, poly) \\ QMIP^*(k, m) &= QMIP(k, m, \infty) \end{aligned}$$

Kobayashi and Matsumoto prove in [10] that

$$QMIP(poly, poly) = MIP(poly, poly) = NEXP$$

Also, they proved that if the provers have $poly(|x|)$ -prior-entanglement then we can assume that $\dim(\mathcal{P}_i) = 2^{poly(|x|)}$ and therefore $QMIP^{poly}(poly, poly) \subseteq QMIP(poly, poly)$. This is not necessarily an equality, because potentially, more entanglement may be used by the provers to cheat the verifier. It is possible that there are languages that can be proved without entanglement and can not be proved with it.

Thus the main difference between the quantum and the classical models is that the provers can use prior-entanglement to their advantage, and otherwise $QMIP = MIP$.

The power of $QMIP^*(poly, poly)$ is a mystery, the provers are stronger and thus it might seem that they may prove more languages. However the provers are also less trustworthy so there might be some languages that the classical provers can prove but quantum provers with entanglement can not. Thus, it is not even known that $QMIP^*(poly, poly) \subseteq NEXP$ or $NEXP \subseteq QMIP^*(poly, poly)$.

In [7] Kempe and Vidick expand the definition of $QMIP(k, 1)$ to $QMIP_{\log n, c, s}(k, 1)$, the class of languages that have a QMIP proof with the verifiers complexity and the message registers logarithmic in the input size. They prove that $NP \subseteq QMIP_{\log n, 1, 1-2^{-o(n)}}^*(2, 1)$. This implies that even if the provers have unlimited entanglement they can not cheat perfectly. Recently, this result have been improved to $1 - \frac{1}{poly(n)}$ soundness. By applying the padding argument this can be expanded to $NEXP \subseteq QMIP_{poly(n), 1, 1-2^{-poly(n)}}^*(2, 1)$.

2.3 The Diamond Norm

In this section we survey Kitaev and Watrous [9] characterization of QIP(3) using the diamond norm.

Definition 2. *The Trace Norm of an operator $A \in L(\mathcal{H})$ is*

$$\|A\|_{\text{tr}} = \max_{U \in U(\mathcal{H})} |\text{Tr}(UA)|$$

If A is a normal matrix with eigenvalues $\{\lambda_i\}$ then $\|A\|_{\text{tr}} = \sum_i |\lambda_i|$. For a general A it can be checked that $\|A\|_{\text{tr}} = \text{Tr}(|A|) = \text{Tr}(\sqrt{AA^\dagger})$. Also $\|A\|_{\text{tr}} = \sum_i s_i(A)$ where $s_1(A) \geq \dots \geq s_n(A)$ are the singular values of A . The natural generalization of the $\|\cdot\|_{\text{tr}}$ to superoperators is

Definition 3. *Let $T : L(\mathcal{H}_1) \rightarrow L(\mathcal{H}_2)$ be a superoperator. The l_1 norm $\|T\|_1$ is*

$$\|T\|_1 = \max_{A: \|A\|_{\text{tr}}=1} \|T(A)\|_{\text{tr}}$$

Definition 4. *A superoperator norm $\|\cdot\|$ is $f(n)$ -stable iff for any $T : L(\mathcal{H}_1) \rightarrow L(\mathcal{H}_2)$ having $\dim(\mathcal{H}_1) = n$ and every $N \geq 0$ it holds that*

$$\|T \otimes I_N\| \leq \|T \otimes I_{f(n)}\|.$$

If $f(n) = 0$ we say that $\|\cdot\|$ is *stable*. The l_1 norm is not stable. For example consider the superoperator on $L(\mathbb{C}^2)$

$$T(|i\rangle\langle j|) = |j\rangle\langle i|, (i, j = 0, 1)$$

On the one hand $\|T\|_1 = 1$. On the other hand for $A = \sum_{i,j} |i, i\rangle\langle j, j|$, $\|A\|_{\text{tr}} = 2$ but $\|T \otimes I_1(A)\|_{\text{tr}} = 4$, and so $\|T \otimes I_1\|_1 \geq 2$.

Fortunately Kitaev [8] proved that $\|\cdot\|_1$ is n -stable. For any $N \geq 0$ and $n = \dim(\mathcal{H}_1)$ it holds that $\|T \otimes I_N\|_1 \leq \|T \otimes I_n\|_1$. Watrous [14] gave a simpler proof of that. This allows to define the diamond norm.

Definition 5. *Let $T : L(\mathcal{H}_1) \rightarrow L(\mathcal{H}_2)$ be a superoperator and $n = \dim(\mathcal{H}_1)$ then the diamond norm $\|T\|_\diamond$ is*

$$\|T\|_\diamond = \|T \otimes I_n\|_1$$

This defines a norm [8]. The $\|\cdot\|_\diamond$ is indeed stable. Kitaev [8] also proved that the diamond norm is multiplicative, i.e., $\|T \otimes R\|_\diamond = \|T\|_\diamond \|R\|_\diamond$. He also gave other equivalent mathematical formulations to it.

2.4 QIP(3) Characterization by the diamond norm

Denote QIP(3, s , c) the class of languages with a QIP proof system with three messages, soundness s and completeness c . Let $L \in \text{QIP}(3, s, 1)$ proved to a verifier V . The protocol is characterized by the unitary operators V_1, V_2 the verifier applies in each round, the initial state projection Π_{init} and the accepting projection Π_{acc} . Denote $B_1 = V_1 \Pi_{\text{init}}, B_2 = \Pi_{\text{acc}} V_2$. Let $\text{MAP}(B_1, B_2)$ denote the maximal acceptance probability of the verifier. Kitaev and Watrous proved that

$$\text{MAP}(B_1, B_2) = \|T\|_\diamond$$

where $T(X) = \text{Tr}_{\mathcal{V}}(B_1 X B_2)$ giving a neat algebraic characterization of the game.

As a corollary of the above characterization and the fact that the diamond norm is multiplicative Kitaev and Watrous showed that QIP(3, s , 1) has perfect parallel amplification.

3 QMIP*(2, 1) and the Product Norm

In this section we define a product operator norm and a product superoperator norm and later prove that the maximum acceptance probability for a given verifier in quantum one round two prover protocol can be described in terms of it.

3.1 The Product Norm

Definition 6. For Hilbert spaces $\mathcal{V}_1, \mathcal{V}_2$ and a matrix $A \in L(\mathcal{V}_1 \otimes \mathcal{V}_2)$ the product norm of A is

$$\|A\|_{\mathcal{V}_1 \otimes \mathcal{V}_2} = \max_{U_i \in U(\mathcal{V}_i)} |\text{Tr}((U_1 \otimes U_2)A)|$$

Claim 1. $\|\cdot\|_{\mathcal{V}_1 \otimes \mathcal{V}_2}$ is a norm.

Proof. The following things are simple.

1. $\|A\|_{\mathcal{V}_1 \otimes \mathcal{V}_2} \geq 0$.
2. $\|cA\|_{\mathcal{V}_1 \otimes \mathcal{V}_2} = |c| \|A\|_{\mathcal{V}_1 \otimes \mathcal{V}_2}$.
3. Triangle inequality.
4. If $A = 0$ then $\|A\|_{\mathcal{V}_1 \otimes \mathcal{V}_2} = 0$.

We are left with showing that if $\|A\|_{\mathcal{V}_1 \otimes \mathcal{V}_2} = 0$ then $A = 0$. Assume $\|A\|_{\mathcal{V}_1 \otimes \mathcal{V}_2} = 0$. Then $\|A\|_{\text{tr}} = \text{Tr}(UA)$ for some $U \in U(\mathcal{V}_1 \otimes \mathcal{V}_2)$. The transformation U can be represented as

$$U = \sum_i a_i (W_i \otimes V_i)$$

where $W_i \in U(\mathcal{V}_1), V_i \in U(\mathcal{V}_2)$. This is true because there is a unitary basis for any $L(\mathcal{H})$. One such possible basis is described in [13]. Thus $\text{Tr}(UA) = \sum_i a_i \text{Tr}((W_i \otimes V_i)A) = 0$ and so $\|A\|_{\text{tr}} = 0$ and $A = 0$. \square

We notice that

$$\|\text{Tr}_{\mathcal{V}_2}(A)\|_{\text{tr}} \leq \|A\|_{\mathcal{V}_1 \otimes \mathcal{V}_2} \leq \|A\|_{\text{tr}} \quad (3)$$

The left inequality follows from Equations (1) and (2) because $\text{Tr}((U_1 \otimes U_2)A) = \text{Tr}(U_1 \text{Tr}_{\mathcal{V}_2}((I \otimes U_2)A))$. The right inequality follows from the fact that $\max_{U_i \in U(\mathcal{V}_i)} |\text{Tr}((U_1 \otimes U_2)A)| \leq \max_{U \in U(\mathcal{V}_1 \otimes \mathcal{V}_2)} |\text{Tr}(UA)|$. Those inequalities can be strict, for example for A of the form $A = |u\rangle\langle v|$. For any such A , $\|A\|_{\text{tr}} = 1$ but we will show later that for $A = |epr\rangle\langle 00|$ it holds that $\|A\|_{\mathbb{C}^2 \otimes \mathbb{C}^2} = \frac{1}{\sqrt{2}}$ (where $|epr\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$). Another example is $A = |00\rangle\langle 11|$ with the partition $\mathcal{V}_1 = \mathcal{V}_2 = \mathbb{C}^2$. On the one hand $\|\text{Tr}_{\mathcal{V}_2}(A)\|_{\text{tr}} = 0$, but as we will show later $\|A\|_{\mathcal{V}_1 \otimes \mathcal{V}_2} = 1$.

3.2 The Superoperator Product Norm

Next, we define a superoperator product norm.

Definition 7. For Hilbert spaces $\mathcal{V}, \mathcal{V}_1, \mathcal{V}_2$ and superoperator $T : L(\mathcal{V}) \rightarrow L(\mathcal{V}_1 \otimes \mathcal{V}_2)$ the superoperator product norm is

$$\|T\|_{\mathcal{V}_1 \otimes \mathcal{V}_2, tr} = \max_{\|A\|_{tr}=1} \|T(A)\|_{\mathcal{V}_1 \otimes \mathcal{V}_2}$$

It is easy to check that this is a norm and that $\|I\|_{\mathcal{V}_1 \otimes \mathcal{V}_2, tr} = 1$. Also, it follows from Equation (3) that $\|T\|_{\mathcal{V}_1 \otimes \mathcal{V}_2, tr} \leq \|T\|_{\diamond}$. A useful fact is:

Claim 2.

$$\|T\|_{\mathcal{V}_1 \otimes \mathcal{V}_2, tr} = \max_{|u\rangle, |v\rangle \in \mathcal{V}} \|T(|u\rangle\langle v|)\|_{\mathcal{V}_1 \otimes \mathcal{V}_2}$$

Proof. Any A satisfying $\|A\|_{tr} = 1$ has a singular value decomposition $A = \sum_i s_i |u_i\rangle\langle v_i|$ for $s_i \geq 0$ and $\sum_i s_i = 1$. Thus

$$\begin{aligned} \|T(A)\|_{\mathcal{V}_1 \otimes \mathcal{V}_2} &= \left\| T\left(\sum_i s_i |u_i\rangle\langle v_i|\right) \right\|_{\mathcal{V}_1 \otimes \mathcal{V}_2} \\ &\leq \sum_i s_i \|T(|u_i\rangle\langle v_i|)\|_{\mathcal{V}_1 \otimes \mathcal{V}_2} \\ &\leq \max_i \|T(|u_i\rangle\langle v_i|)\|_{\mathcal{V}_1 \otimes \mathcal{V}_2} \end{aligned}$$

Thus the maximum is always achieved on some rank one matrix $|u\rangle\langle v|$. \square

Claim 3. For any two superoperators $T : L(\mathcal{H}_1) \rightarrow L(\mathcal{V}_1 \otimes \mathcal{V}_2)$ and $R : L(\mathcal{H}_2) \rightarrow L(\mathcal{W}_1 \otimes \mathcal{W}_2)$ it holds that

$$\|T \otimes R\|_{(\mathcal{V}_1 \otimes \mathcal{W}_1) \otimes (\mathcal{V}_2 \otimes \mathcal{W}_2), tr} \geq \|T\|_{\mathcal{V}_1 \otimes \mathcal{V}_2, tr} \cdot \|R\|_{\mathcal{W}_1 \otimes \mathcal{W}_2, tr}$$

Proof.

$$\|T \otimes R\|_{(\mathcal{V}_1 \otimes \mathcal{W}_1) \otimes (\mathcal{V}_2 \otimes \mathcal{W}_2), tr} = \max_{\|X\|_{tr}=1} \|(T \otimes R)(X)\|_{(\mathcal{V}_1 \otimes \mathcal{W}_1) \otimes (\mathcal{V}_2 \otimes \mathcal{W}_2)}$$

Let us look at the special case where $X \in L(\mathcal{H}_1 \otimes \mathcal{H}_2)$ is product, $X = A \otimes B$ for some $A \in L(\mathcal{H}_1)$ and $B \in L(\mathcal{H}_2)$.

$$\begin{aligned} \|T \otimes R\|_{(\mathcal{V}_1 \otimes \mathcal{W}_1) \otimes (\mathcal{V}_2 \otimes \mathcal{W}_2), tr} &\geq \max_{\|A\|_{tr}=\|B\|_{tr}=1} \|(T \otimes R)(A \otimes B)\|_{(\mathcal{V}_1 \otimes \mathcal{W}_1) \otimes (\mathcal{V}_2 \otimes \mathcal{W}_2)} \\ &= \max_{\|A\|_{tr}=\|B\|_{tr}=1} \|T(A) \otimes R(B)\|_{(\mathcal{V}_1 \otimes \mathcal{W}_1) \otimes (\mathcal{V}_2 \otimes \mathcal{W}_2)} \\ &= \max_{\|A\|_{tr}=\|B\|_{tr}=1, U_1, U_2} \text{Tr}((U_1 \otimes U_2)(T(A) \otimes R(B))) \end{aligned}$$

for unitaries $U_1 \in U(\mathcal{V}_1 \otimes \mathcal{W}_1)$ and $U_2 \in U(\mathcal{V}_2 \otimes \mathcal{W}_2)$. We again look at the special case where U_1 and U_2 are also products of unitaries $U_1 = V_1 \otimes W_1$ and $U_2 = V_2 \otimes W_2$ for $V_1 \in U(\mathcal{V}_1), W_1 \in U(\mathcal{W}_1), V_2 \in U(\mathcal{V}_2), W_2 \in U(\mathcal{W}_2)$. Then

$$\begin{aligned}
& \|T \otimes R\|_{(\mathcal{V}_1 \otimes \mathcal{W}_1) \otimes (\mathcal{V}_2 \otimes \mathcal{W}_2), tr} \\
& \geq \max_{\|A\|_{tr} = \|B\|_{tr} = 1, V_1, V_2, W_1, W_2} \text{Tr}((V_1 \otimes W_1 \otimes V_2 \otimes W_2)(T(A) \otimes R(B))) \\
& = \max_{\|A\|_{tr} = \|B\|_{tr} = 1, V_1, V_2, W_1, W_2} \text{Tr}((V_1 \otimes V_2)T(A)) \cdot \text{Tr}((W_1 \otimes W_2)R(B)) \\
& = \|T\|_{\mathcal{V}_1 \otimes \mathcal{V}_2, tr} \cdot \|R\|_{\mathcal{W}_1 \otimes \mathcal{W}_2, tr}
\end{aligned}$$

□

In particular it follows from above that

$$\|T\|_{\mathcal{V}_1 \otimes \mathcal{V}_2, tr} \leq \|T \otimes I_{\mathcal{W}_1 \otimes \mathcal{W}_2}\|_{(\mathcal{V}_1 \otimes \mathcal{W}_1) \otimes (\mathcal{V}_2 \otimes \mathcal{W}_2), tr}$$

Next we expand the definition of stability to the superoperator product norm. We do this by adding to each register of the original partition $\mathcal{V}_1, \mathcal{V}_2$ an additional register \mathbb{C}^N and applying the superoperator $T \otimes I_N \otimes I_N$ with the identity operator over the new registers.

Definition 3.1. $A\|_{\mathcal{V}_1 \otimes \mathcal{V}_2, tr}$ is $f(n)$ -stable iff for any $T : L(\mathcal{H}) \rightarrow L(\mathcal{V}_1 \otimes \mathcal{V}_2)$ having $\dim(\mathcal{H}) = n$ and every $N \geq 0$ it holds that

$$\|T \otimes I_{N^2}\|_{(\mathcal{V}_1 \otimes \mathbb{C}^N) \otimes (\mathcal{V}_2 \otimes \mathbb{C}^N), tr} \leq \|T \otimes I_{f(n)^2}\|_{(\mathcal{V}_1 \otimes \mathbb{C}^{f(n)}) \otimes (\mathcal{V}_2 \otimes \mathbb{C}^{f(n)}), tr}$$

The $\|_{\mathcal{V}_1 \otimes \mathcal{V}_2, tr}$ norm is not 0 stable. Consider the superoperator $T : L(\mathbb{C}^4) \rightarrow L(\mathbb{C}^2 \otimes \mathbb{C}^2)$ that is defined by $T(|i, j\rangle\langle k, m|) = |k, m\rangle\langle i, j|$. Then $\|T\|_{\mathbb{C}^2 \otimes \mathbb{C}^2, tr} \leq \|T\|_1 = 1$. On the other hand, $\|T \otimes I_4\|_{(\mathbb{C}^2 \otimes \mathbb{C}^2) \otimes (\mathbb{C}^2 \otimes \mathbb{C}^2), tr} = 4$. To see that use $A = \sum_{i,j,k,m} |i, j, i, j\rangle\langle k, m, k, m|$. It is easy to check that $\|A\|_{tr} = 4$, and that by $U|i, k\rangle = |k, i\rangle$ we have $(U \otimes U)(T \otimes I_4)(A) = \sum_{i,j,k,m} |i, k, j, m\rangle\langle i, k, j, m| = I_{16}$ and so $\|T \otimes I_4\|_{(\mathbb{C}^2 \otimes \mathbb{C}^2) \otimes (\mathbb{C}^2 \otimes \mathbb{C}^2), tr} \geq 4$. Altogether $\|T \otimes I_4\|_{(\mathbb{C}^2 \otimes \mathbb{C}^2) \otimes (\mathbb{C}^2 \otimes \mathbb{C}^2), tr} \leq \|T\|_{\diamond} = 4$.

3.3 QMIP*(2, 1)

In this section we focus on QMIP*(2, 1). The protocol is applied on the registers $\mathcal{V} \otimes \mathcal{M}_1 \otimes \mathcal{M}_2 \otimes \mathcal{P}_1 \otimes \mathcal{P}_2$ where \mathcal{V} is the verifier's private register. $\mathcal{M}_1, \mathcal{M}_2$ are the registers passed between V and P_1, P_2 respectively. $\mathcal{P}_1, \mathcal{P}_2$ are the private registers of the provers. The initial quantum state is some $|\psi\rangle$ of an arbitrary length chosen as part of the prover strategy.

The protocol proceeds as follows:

1. The verifier applies a measurement defined by $\Pi_{init} = |0\rangle\langle 0|$ on $\mathcal{V} \otimes \mathcal{M}_1 \otimes \mathcal{M}_2$. If the outcome is not $|0\rangle$ he rejects. This step checks the initial state.
2. The verifier applies a unitary transformation V_1 on $\mathcal{V} \otimes \mathcal{M}_1 \otimes \mathcal{M}_2$. This prepares the questions to the two provers.
3. Prover i applies a unitary U_i on $\mathcal{M}_i \otimes \mathcal{P}_i$.
4. The verifier applies a unitary V_2 on $\mathcal{V} \otimes \mathcal{M}_1 \otimes \mathcal{M}_2$, followed by a measurement defined by $\Pi_{acc} = |0\rangle\langle 0|$ on the first qubit of \mathcal{V} and accepts iff the outcome is $|0\rangle$.

If the provers are successful in convincing the verifier the final (unnormalized) state of the system is thus

$$((\Pi_{acc} V_2) \otimes I_{\mathcal{P}_1 \otimes \mathcal{P}_2})(I_{\mathcal{V}} \otimes U_1 \otimes U_2)((V_1 \Pi_{init}) \otimes I_{\mathcal{P}_1 \otimes \mathcal{P}_2})|\psi\rangle$$

3.4 Acceptance Probability for a Given Verifier

Let V be a verifier. V 's strategy is defined by $B_1 = V_1 \Pi_{init}$ and $B_2 = \Pi_{acc} V_2$. Let $\text{MAP}(B_1, B_2)$ denote the maximum acceptance probability of V , when V plays with the optimal provers. I.e.,

$$\text{MAP}(B_1, B_2) = \max_{U_i \in U(\mathcal{M}_i \otimes \mathcal{P}_i), |\psi\rangle} |(B_2 \otimes I_{\mathcal{P}_1 \otimes \mathcal{P}_2})(I_{\mathcal{V}} \otimes U_1 \otimes U_2)(B_1 \otimes I_{\mathcal{P}_1 \otimes \mathcal{P}_2})|\psi\rangle|^2 \quad (4)$$

We now relate $\text{MAP}(B_1, B_2)$ to the superoperator product norm. We claim that:

Theorem 3.2. $\text{MAP}(B_1, B_2) = \|T \otimes I_{\mathcal{P}_1 \otimes \mathcal{P}_2}\|_{(\mathcal{M}_1 \otimes \mathcal{P}_1) \otimes (\mathcal{M}_2 \otimes \mathcal{P}_2), tr}^2$

where $T : L(\mathcal{V} \otimes \mathcal{M}_1 \otimes \mathcal{M}_2) \rightarrow L(\mathcal{M}_1 \otimes \mathcal{M}_2)$ is defined by $T(X) = \text{Tr}_{\mathcal{V}}(B_1 X B_2)$.

Proof. Denote $\mathcal{P} = \mathcal{P}_1 \otimes \mathcal{P}_2$. We start with Equation (4).

$$\sqrt{\text{MAP}(B_1, B_2)} = \max_{U_1, U_2, \psi} |(B_2 \otimes I_{\mathcal{P}})(I_{\mathcal{V}} \otimes U_1 \otimes U_2)(B_1 \otimes I_{\mathcal{P}})|\psi\rangle|$$

Since we maximize over the unit vector $|\psi\rangle$ we can replace the vector norm with the operator norm

$$\sqrt{\text{MAP}(B_1, B_2)} = \max_{U_1, U_2} \|(B_2 \otimes I_{\mathcal{P}})(I_{\mathcal{V}} \otimes U_1 \otimes U_2)(B_1 \otimes I_{\mathcal{P}})\|$$

The operator norm of the matrix is the largest singular value, and so

$$\sqrt{\text{MAP}(B_1, B_2)} = \max_{U_1, U_2, v, u} |\langle v | (B_2 \otimes I_{\mathcal{P}})(I_{\mathcal{V}} \otimes U_1 \otimes U_2)(B_1 \otimes I_{\mathcal{P}}) |u\rangle|$$

Since this is a scalar number we can insert trace

$$\begin{aligned} \sqrt{\text{MAP}(B_1, B_2)} &= \max_{U_1, U_2, v, u} |\text{Tr}(\langle v | (B_2 \otimes I_{\mathcal{P}})(I_{\mathcal{V}} \otimes U_1 \otimes U_2)(B_1 \otimes I_{\mathcal{P}}) |u\rangle)| \\ &= \max_{U_1, U_2, v, u} |\text{Tr}((I_{\mathcal{V}} \otimes U_1 \otimes U_2)(B_1 \otimes I_{\mathcal{P}}) |u\rangle \langle v | (B_2 \otimes I_{\mathcal{P}}))| \end{aligned}$$

By Equation (1)

$$\sqrt{\text{MAP}(B_1, B_2)} = \max_{U_1, U_2, v, u} |\text{Tr}(\text{Tr}_{\mathcal{V}}((I_{\mathcal{V}} \otimes U_1 \otimes U_2)(B_1 \otimes I_{\mathcal{P}}) |u\rangle \langle v | (B_2 \otimes I_{\mathcal{P}})))|$$

By Equation (2) we can carry the operators that do not affect \mathcal{V} out, use the definition of T and then use Claim 2.

$$\begin{aligned} \sqrt{\text{MAP}(B_1, B_2)} &= \max_{U_1, U_2, u, v} |\text{Tr}((U_1 \otimes U_2) \text{Tr}_{\mathcal{V}}((B_1 \otimes I_{\mathcal{P}}) |u\rangle \langle v | (B_2 \otimes I_{\mathcal{P}})))| \\ &= \max_{U_1, U_2, u, v} |\text{Tr}((U_1 \otimes U_2)(T \otimes I_{\mathcal{P}})(|u\rangle \langle v|))| \\ &= \|T \otimes I_{\mathcal{P}}\|_{(\mathcal{M}_1 \otimes \mathcal{P}_1) \otimes (\mathcal{M}_2 \otimes \mathcal{P}_2), tr} \end{aligned}$$

□

Let us notice that this proof is almost identical to the proof of QIP(3) characterization by Kitaev and Watrous [9]. The main difference is that here we have a product norm instead of the trace norm as a target. This is because the initial state of the provers in $\text{QMIP}^*(2, 1)$ can be viewed as the first message and so we actually have three messages instead of two.

4 Product Norm of Rank 1 Matrices

We start with a useful bound on $\|BC\|_{\text{tr}}$ and use it to show what is the product norm for rank 1 matrices.

Lemma 4.1. *Fix arbitrary matrices B and C with $s_1(B) \geq \dots \geq s_n(B) \geq 0$ the singular values of B , and $s_1(C) \geq \dots \geq s_n(C) \geq 0$ the singular values of C . Then*

$$\|BC\|_{\text{tr}} \leq \sum_i s_i(B)s_i(C)$$

The above claim appears in [6] (page 182, Exercise 4). Notice also that this is tight for normal commuting matrices B and C .

With that we prove:

Theorem 4.2. *Let A be a rank 1 matrix over $\mathcal{V}_1 \otimes \mathcal{V}_2$. Thus $A = |u\rangle\langle v|$ for some $u, v \in \mathcal{V}_1 \otimes \mathcal{V}_2$. Suppose the Schmidt decomposition of u is $|u\rangle = \sum_i \alpha_i |x_i\rangle \otimes |y_i\rangle$, and of v is $|v\rangle = \sum_i \beta_i |w_i\rangle \otimes |z_i\rangle$ with $\alpha_i, \beta_i \geq 0$ sorted in descending order. Then*

$$\|A\|_{\mathcal{V}_1 \otimes \mathcal{V}_2} = \sum_i \alpha_i \beta_i$$

Proof. We can assume without loss of generality that $|x_i\rangle = |y_i\rangle = |w_i\rangle = |z_i\rangle = |i\rangle$ because $\|A\|_{\mathcal{V}_1 \otimes \mathcal{V}_1} = \|(U_1 \otimes U_2)A(V_1 \otimes V_2)\|_{\mathcal{V}_1 \otimes \mathcal{V}_1}$ for any unitaries $U_1, V_1 \in U(\mathcal{V}_1)$ and $U_2, V_2 \in U(\mathcal{V}_2)$. Thus

$$A = |u\rangle\langle v| = \sum_{i,j} \alpha_i \beta_j |i, i\rangle\langle j, j|$$

and

$$\begin{aligned} \text{Tr}((U_1 \otimes U_2)A) &= \sum_{i,j} \alpha_i \beta_j \langle j| U_1 |i\rangle \langle j| U_2 |i\rangle \\ &= \sum_{i,j} \alpha_i (U_1)_{j,i} \cdot \beta_j (U_2)_{j,i} \end{aligned}$$

We can look at this sum of products as a standard matrix inner product. Let us denote the matrices C and B as follows, $C_{j,i} = \alpha_i (U_1)_{j,i}$ and $B_{j,i} = \beta_j (U_2)_{j,i}$. Then

$$\text{Tr}((U_1 \otimes U_2)A) = \sum_{i,j} B_{i,j} C_{i,j} = \text{Tr}(B^t C)$$

By Lemma 4.1, $|\text{Tr}(B^t C)| \leq \|B^t C\|_{\text{tr}} \leq \sum_i \alpha_i \beta_i$, because $C = U_1 \text{diag}(\alpha_1, \dots, \alpha_n)$, $B = \text{diag}(\beta_1, \dots, \beta_n) U_2$, and so $s_i(C) = \alpha_i$ and $s_i(B) = \beta_i$. Finally, this upper bound can be achieved by $U_1 = U_2 = I$. \square

5 Directions for Further Research

We can *not* prove that the product norm stabilizes. However we would like to check what such a result would give.

Hypothesis 1. $\|\cdot\|_{\mathcal{V}_1 \otimes \mathcal{V}_2, \text{tr}}$ is *poly*(n)-stable.

Claim 4. *Under hypothesis 1 $QMIP^*(2, 1) \subseteq NEXP = MIP(2, 1)$.*

Proof. Let $L \in QMIP^*(2, 1)$. Consider a verifier V for L . By Theorem 5.1 the maximum acceptance probability of V is

$$\text{MAP}(B_1, B_2) = \|T \otimes I_{\mathcal{P}_1 \otimes \mathcal{P}_2}\|_{(\mathcal{M}_1 \otimes \mathcal{P}_1) \otimes (\mathcal{M}_2 \otimes \mathcal{P}_2), tr}^2$$

for B_1, B_2 and T defined as before. It follows from Definitions 6,7 and Claim 2 that

$$\text{MAP}(B_1, B_2) = \text{Tr}((U_1 \otimes U_2)(T \otimes I_{\mathcal{P}_1 \otimes \mathcal{P}_2})(|u\rangle\langle v|))$$

for some $U_1 \in U(\mathcal{M}_1 \otimes \mathcal{P}_1)$, $U_2 \in U(\mathcal{M}_2 \otimes \mathcal{P}_2)$ and $|u\rangle, |v\rangle \in \mathcal{V} \otimes \mathcal{M}_1 \otimes \mathcal{M}_2 \otimes \mathcal{P}_1 \otimes \mathcal{P}_2$. Under the hypothesis we can fix such U_1, U_2 and $|u\rangle, |v\rangle$ that live in the world of $poly(|x|)$ qubits. Consider the prover strategy $U_1 \otimes U_2$ with the initial state $|u\rangle$. This strategy uses only $poly(|x|)$ entangled qubits in the initial state and is optimal. Thus $QMIP^*(2, 1) \subseteq QMIP^{poly}(2, 1)$ and we already mentioned that Kobayashi and Matsumoto proved in [10] that $QMIP^{poly}(2, 1) \subseteq NEXP$. \square

Another hypothesis is the following. It is not known if there exists an efficient Turing machine for approximating the $\|\cdot\|_{\diamond}$. However Kitaev and Watrous proved in [9] that $QIP \subseteq EXP$ by showing a reduction from distinguishing between the case of $\text{MAP}(B_1, B_2) = 1$ and $\text{MAP}(B_1, B_2) \leq \frac{1}{2}$ to a semidefinite programming problem of an exponential size (in the number of qubits).

Hypothesis 2. *For $T : L(\mathcal{H}) \rightarrow L(\mathcal{V}_1 \otimes \mathcal{V}_2)$ there exists a Turing machine that approximates $\|T\|_{\mathcal{V}_1 \otimes \mathcal{V}_2, tr}$ in $poly(\dim(\mathcal{H}) + \dim(\mathcal{V}_1 \otimes \mathcal{V}_2))$ time.*

Claim 5. *If both hypotheses are true then $QMIP^*(2, 1) \subseteq EXP$.*

Proof. Let $L \in QMIP^*(2, 1)$. Hypothesis 1 implies that L has a protocol $\langle V, P_1, P_2 \rangle$ with maximum acceptance probability

$$\|T \otimes I_{\mathcal{P}_1 \otimes \mathcal{P}_2}\|_{(\mathcal{M}_1 \otimes \mathcal{P}_1) \otimes (\mathcal{M}_2 \otimes \mathcal{P}_2), tr}^2$$

for T defined as previously and $\dim(\mathcal{M}_1 \otimes \mathcal{P}_1 \otimes \mathcal{M}_2 \otimes \mathcal{P}_2) = 2^{poly(|x|)}$. Hypothesis 2 implies that there is a Turing machine that approximates the maximum acceptance probability and decides if $x \in L$ in $poly(2^{poly(|x|)})$ time. \square

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