

# Symmetric Logspace is Closed Under Complement \*

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## Abstract

We present a Logspace, many-one reduction from the undirected st-connectivity problem to its complement. This shows that  $SL = co - SL$ .

## 1 Introduction

This paper deals with the complexity class symmetric Logspace,  $SL$ , defined by Lewis and Papadimitriou in [LP82]. This class can be defined in several equivalent ways:

1. Languages which can be recognised by symmetric nondeterministic Turing

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Machines that run within logarithmic space. See [LP82].

2. Languages that can be accepted by a uniform family of polynomial size contact schemes (also sometimes called switching networks.) See [Raz91].
3. Languages which can be reduced in Logspace via a many-one reduction to  $USTCON$ , the undirected st-connectivity problem.

A major reason for the interest in this class is that it captures the complexity of  $USTCON$ . The input to  $USTCON$  is an undirected graph  $G$  and two vertices in it  $s, t$ , and the input should be accepted if  $s$  and  $t$  are connected via a path in  $G$ . The similar problem,  $STCON$ , where the graph  $G$  is allowed to be directed is complete for  $NL$ , non-deterministic Logspace. Several combinatorial problems are known to be in  $SL$  or  $co - SL$ , e.g. 2-colourability is complete in  $co - SL$  [Rei82].

The following facts are known regarding  $SL$  relative to other complexity classes in “the vicinity”:

$$L \subseteq SL \subseteq RL \subseteq NL.$$

Here,  $L$  is the class deterministic Logspace and  $RL$  is the class of problems that can be accepted with one-sided error by a randomized Logspace machine running in polynomial time. The containment  $SL \subseteq$

$RL$  is the only non-trivial one in the line above and follows directly from the randomized Logspace algorithm for  $USTCON$  of [AKL<sup>+</sup>79]. It is also known that  $SL \subseteq SC$  [Nis92],  $SL \subseteq \bigoplus L$  [KW93] and  $SL \subseteq DSPACE(\log^{1.5} n)$  [NSW92].

After the surprising proofs that  $NL$  is closed under complement were found [Imm88, Sze88], Borodin et al [BCD<sup>+</sup>89] asked whether the same is true for  $SL$ . They could prove only the weaker statement, namely that  $SL \subseteq co - RL$ , and left “ $SL = co - SL$ ?” as an open problem. In this paper we solve the problem in the affirmative by exhibiting a Logspace, many-one reduction from  $USTCON$  to its complement. Quite surprisingly the proof of our theorem does not use inductive counting, as do the proofs of  $NL = co - NL$ , and is in fact even simpler than them, however it uses the [AKS83] sorting networks.

**Theorem 1**  $SL = co - SL$ .

It should be noted that the monotone analogues (see [GS91]) of  $SL$  and  $co - SL$  are known to be different [KW88].

As a direct corollary of our theorem, we get that  $L^{<SL>} = SL$  where  $L^{<SL>}$  is the class of languages accepted by *Logspace* oracle Turing machines with oracle from  $SL$ , being careful with the way we allow queries (see [RST84]).

**Corollary 1.1**  $L^{<SL>} = SL$

In particular we show that both “symmetric Logspace hierarchies”, (the one defined by alternation in [Rei82], and the one defined by oracle queries in [BPS92]) collapse to  $SL$ .

## 2 Proof of Theorem

### 2.1 Overview of proof.

We design a many-one reduction from  $co - USTCON$  to  $USTCON$ . We start by developing, in subsection 2.2, simple tools for combining reductions. In particular these tools will allow us to use the AKS sorting networks in order to “count”. At this point, the main ingredient of the reduction will be the calculation of the number of the connected components of a graph. An upper bound to this number is easily obtained using transitive closure, while the main idea of the proof is to obtain a lower bound by computing a spanning forest of the graph, which is done in subsection 2.3. In subsection 2.4 everything is put together.

### 2.2 Projections to $USTCON$ .

In this paper we will use only the simplest kind of reductions, i.e. *LogSpace* uniform projection reductions [SV85]. Moreover, we will be interested only in reductions to  $USTCON$ . In this subsection we define this kind of reduction and we show some of its basic properties.

**NOTATION 2.1** Given  $f : \{0,1\}^* \mapsto \{0,1\}^*$  denote by  $f_n : \{0,1\}^n \mapsto \{0,1\}^*$  the restriction of  $f$  to inputs of length  $n$ . Denote by  $f_{n,k}$  the  $k$ 'th bit function of  $f_n$ , i.e. if  $f_n : \{0,1\}^n \mapsto \{0,1\}^{k(n)}$  then  $f_n = (f_{n,1}, \dots, f_{n,k(n)})$ .

**NOTATION 2.2** We represent an  $n$ -node undirected graph  $G$  using  $\binom{n}{2}$  variables  $\vec{x} = \{x_{i,j}\}_{1 \leq i < j \leq n}$  s.t.  $x_{i,j}$  is 1 iff  $(i,j) \in E(G)$ . If  $f(\vec{x})$  operates on graphs, we will write  $f(G)$  meaning that the input to  $f$  is a binary vector of length  $\binom{n}{2}$  representing  $G$ .

We say that  $f : \{0,1\}^* \mapsto \{0,1\}^*$  reduces to  $USTCON(m)$  if we can (uniformly

and in *LogSpace* ) label the edges of a graph of size  $m$  with  $\{0, 1, x_i, \neg x_i\}_{1 \leq i \leq n}$ , s.t.  $f_{n,k}(\vec{x}) = 1 \iff$  there is a path from 1 to  $m$  in the corresponding graph. Formally, □

**DEFINITION 2.1** We say that  $f : \{0, 1\}^* \mapsto \{0, 1\}^*$  reduces to  $USTCON(m)$ ,  $m = m(n)$ , if there is a uniform family of  $Space(\log(n))$  functions  $\{\sigma_{n,k}\}$  s.t. for all  $n$  and  $k$ :

- $\sigma_{n,k}$  is a projection. i.e.:  $\sigma_{n,k}$  is a mapping from  $\{i, j\}_{1 \leq i < j \leq m}$  to  $\{0, 1, x_i, \neg x_i\}_{1 \leq i \leq n}$
- Given  $\vec{x}$  define  $G_{\vec{x},k}$  to be the graph  $G_{\vec{x},k} = (\{1, \dots, m\}, E)$  where  $E = \{(i, j) \mid \sigma_{n,k}(i, j) = 1 \text{ or } \sigma_{n,k}(i, j) = x_i \text{ and } x_i = 1 \text{ or } \sigma_{n,k}(i, j) = \neg x_i \text{ and } x_i = 0\}$ .
- $f_{n,k}(\vec{x}) = 1 \iff$  there is a path from 1 to  $m$  in  $G_{\vec{x},k}$ .

If  $\sigma$  is restricted to the set  $\{0, 1, x_i\}_{1 \leq i \leq n}$  we say that  $f$  monotonically reduces to  $USTCON(m)$ .

**Lemma 2.1** If  $f$  has uniform monotone formulae of size  $s(n)$  then  $f$  is monotonically reducible to  $USTCON(O(s(n)))$ .

**Proof:** Given a formula  $\phi$  recursively build  $(G, s, t)$  as follows:

- If  $\phi = x_i$  then build a graph with two vertices  $s$  and  $t$ , and one edge between them labeled with  $x_i$ .
- If  $\phi = \phi_1 \wedge \phi_2$ , and  $(G_i, s_i, t_i)$  the graphs for  $\phi_i$ ,  $i = 1, 2$ , then identify  $s_2$  with  $t_1$  and define  $s = s_1, t = t_2$ .
- If  $\phi = \phi_1 \vee \phi_2$ , and  $(G_i, s_i, t_i)$  the graphs for  $\phi_i$ ,  $i = 1, 2$ , then identify  $s_1$  with  $t_1$  and  $s_2$  with  $t_2$  and define  $s = s_1 = t_1$  and  $t = s_2 = t_2$ .

**DEFINITION 2.2**  $Sort : \{0, 1\}^n \mapsto \{0, 1\}^n$  is the boolean sorting function, i.e. it moves all the zeroes to the beginning of the string.

Using the AKS sorting networks [AKS83], which belong to  $NC^1$ , we get:

**Corollary 2.2**  $Sort$  is monotonically reducible to  $USTCON(poly)$ .

**Lemma 2.3** If

$f$  monotonically reduces to  $USTCON(m_1)$  and  $g$  reduces to  $USTCON(m_2)$  then  $f \circ g$  reduces to  $USTCON(m_1^2 \cdot m_2)$ , where  $\circ$  is the standard function composition operator.

**Proof:**  $f$  monotonically reduces to a graph with  $m_1$  vertices, where each edge is labeled with one of  $\{0, 1, x_i\}$ . In the composition function  $f \circ g$  each  $x_i$  is replaced by  $x_i = g_i(\vec{y})$  which can be reduced to a connectivity problem of size  $m_2$ . Replace each edge labeled  $x_i$  with its corresponding connectivity problem. There can be  $m_1^2$  edges, each replaced by a graph with  $m_2$  vertices, hence the new graph has  $m_1^2 \cdot m_2$  vertices. □

### 2.3 Finding a spanning forest.

In this section we show how to build a spanning forest using  $USTCON$ . This basic idea was already noticed by Reif and independently by Cook [Rei82].

Given a graph  $G$  index the edges from 1 to  $m$ . We can view the indices as weights to the edges, and as no two edges have the same weight, we know that there is a unique minimal spanning forest  $F$ . In our case, where the edges are indexed, this minimal forest is the lexicographically first spanning forest.

It is well known that the greedy algorithm finds a minimal spanning forest. Let us recall how the greedy algorithm works in our case. The algorithm builds a spanning forest  $F$  which is at the beginning empty  $F = \emptyset$ . Then the algorithm checks the edges one by one according to their order, for each edge  $e$  if  $e$  does not close a cycle in  $F$  then  $e$  is added to the forest, i.e.  $F = F \cup \{e\}$ .

At first glance the algorithm looks sequential, however, claim 2.3 shows that the greedy algorithm is actually highly parallel. Moreover, all we need to check that an edge does not participate in the forest, is one *st* connectivity problem over a simple to get graph.

**DEFINITION 2.3** For an undirected graph  $G$  denote by  $LEFF(G)$  the lexicographically first spanning forest of  $G$ . Let

$$SF(G) \mapsto \{0, 1\}^{\binom{n}{2}} \text{ be:}$$

$$SF_{i,j}(G) = \begin{cases} 0 & (i, j) \in LEFF(G) \\ 1 & \text{otherwise} \end{cases}$$

**Lemma 2.4**  $SF$  reduces to  $USTCON(\text{poly})$

**Proof:** Let  $F$  be the lexicographically first spanning forest of  $G$ . For  $e \in E$  define  $G_e$  to be the subgraph of  $G$  containing only the edges  $\{e' \in E \mid \text{index}(e') < \text{index}(e)\}$ .

**Claim:**  $e = (i, j) \in F \iff e \in E$  and  $i$  is not connected to  $j$  in  $G_e$ .

**Proof:** Let  $e = (i, j) \in E$ . Denote by  $F_e$  the forest which the greedy algorithm built at the time it was checking  $e$ . So  $e \in F \iff e$  does not close a cycle in  $F_e$ .

$(\implies)$   $e \in F$  and therefore  $e$  does not close a cycle in  $F_e$ , but then  $e$  does not close a cycle in the transitive closure of  $F_e$ , and in particular  $e$  does not close a cycle in  $G_e$ .

$(\impliedby)$   $e$  does not close a cycle in  $G_e$  therefore  $e$  does not close a cycle in  $F_e$  and  $e \in F$ .  $\square$

Therefore  $SF_{i,j}(G) = \neg x_{i,j} \vee i$  is connected to  $j$  in  $G_{(i,j)}$ .

Since  $\neg x_{i,j}$  can be viewed as the connectivity problem over the graph with two vertices and one edge labeled  $\neg x_{i,j}$  it follows from lemmas 2.1, 2.3 that  $SF$  reduces to  $USTCON$ . Notice, however, that the reduction is not monotone.  $\square$

## 2.4 Putting it together.

First, we want to build a function that takes one representative from each connected component. We define  $LI_i(G)$  to be 0 iff the vertex  $i$  has the largest index in its connected component.

**DEFINITION 2.4**  $LI(G) \mapsto \{0, 1\}^n$

$$LI_i(G) = \begin{cases} 0 & i \text{ has the largest index} \\ & \text{in its connected component} \\ 1 & \text{otherwise} \end{cases}$$

**Lemma 2.5**  $LI$  reduces to  $USTCON(\text{poly})$

**Proof:**

$LI_i(G) = \bigvee_{j=i+1}^n (i \text{ is connected to } j \text{ in } G)$ .

So  $LI$  is a simple monotone formula over connectivity problems, and by lemmas 2.1, 2.3  $LI$  reduces to  $USTCON$ . This is, actually, a monotone reduction.  $\square$

Using the spanning forest and the  $LI$  function we can exactly compute the number of connected components of  $G$ , i.e.:

given  $G$  we can compute a function  $NCC_i$  which is 1 iff there are exactly  $i$  connected components in  $G$ .

DEFINITION 2.5  $NCC(G) \mapsto \{0, 1\}^n$

$$NCC_i(G) = \begin{cases} 1 & \text{there are exactly } i \\ & \text{connected components} \\ & \text{in } G \\ 0 & \text{otherwise} \end{cases}$$

**Lemma 2.6**  $NCC$  reduces to  $USTCON(poly)$

**Proof:**

Let  $F$  be a spanning forest of  $G$ . It is easy to see that if  $G$  has  $k$  connected components then  $|F| = n - k$ .

Define:

$$\begin{aligned} f(G) &= Sort \circ LI(G) \\ g(G) &= Sort \circ SF(G). \end{aligned}$$

Then:

$$\begin{aligned} f_i(G) = 1 &\implies k < i \\ g_i(G) = 1 &\implies n - k < i \implies k > n - i. \end{aligned}$$

and thus:  $NCC_i(G) = f_{i+1}(G) \wedge g_{n-i+1}(G)$

Therefore applying lemmas 2.1, 2.2, 2.3, 2.4, 2.5 proves the lemma.  $\square$

Finally we can reduce the non-connectivity problem to the connectivity problem, thus proving that  $SL = co - SL$ .

**Lemma 2.7**  $\overline{USTCON}$  reduces to  $USTCON(poly)$

**Proof:**

Given  $(G, s, t)$  define  $G^+$  to be the graph  $G \cup \{(s, t)\}$ .

Denote by  $\#CC(H)$  the number of connected components in the undirected graph  $H$ .

$s$  is not connected to  $t$  in  $G \iff$

$$\#CC(G^+) = \#CC(G) - 1 \iff$$

$$\bigvee_{i=2, \dots, n} NCC_i(G) \wedge NCC_{i-1}(G^+).$$

Therefore applying lemmas 2.1, 2.3, 2.6 proves the lemma.  $\square$

### 3 Extensions

Denote by  $L^{<SL>}$  the class of languages accepted by *Logspace* oracle Turing machines with an oracle from  $SL$ . An oracle Turing machine has a work tape and a write-only query tape (with unlimited length) which is initialised after every query. We get:

**Corollary 3.1**  $L^{<SL>} = SL$ .

**Proof:**

Let  $Lang$  be a language in  $L^{<SL>}$  computed by an oracle Turing machine  $M$  running in  $L^{<SL>}$ , and fix an input  $\vec{x}$  to  $M$ .

We build the ‘‘configuration’’ graph  $G(V, E)$  of  $M$ , by:

- Let  $V$  contain all possible configurations.
- $(v, w) \in E$  with the label ‘‘ $q$  is (not)  $s$ - $t$  connected’’, if starting from configuration  $v$ , the next query is  $q$ , and when the oracle answers that ‘‘ $q$  is (not) connected’’ then the machine moves to configuration  $w$ .

Notice that we can ignore the direction of the edges, as backward edges do not benefit us. The reason is that from any vertex  $v$ ,

there is only one forward edge leaving  $v$  that can be traversed (i.e. whose label matches the oracle's answer). Therefore if we reach  $v$  using a "backward edge"  $w \mapsto v$ , then the only forward edge leaving  $v$  that can be traversed is  $v \mapsto w$ .

Now we can replace query edges labeled " $q$  is connected" with the  $s$ - $t$  connectivity problem  $q$ , and edges labeled " $q$  is not connected" with the  $s$ - $t$  connectivity problem obtained using our theorem that  $SL = co - SL$ , resulting in one, not too big,  $s$ - $t$  connectivity problem. It is also clear that this can be done in *LogSpace*, completing the proof. □

As the symmetric Logspace hierarchy defined in [Rei82] is known to be within  $L^{<SL>}$ , this hierarchy collapses to  $SL$ .

As can easily be seen, the above argument holds for any undirected graph with undirected query edges, which is exactly the definition of  $SL^{<SL>}$  given by [BPS92]. Thus,  $SL^{<SL>} = SL$ , and by induction the  $SL$  hierarchy defined in [BPS92] collapses to  $SL$ .

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