Abstract

In 2007 Guruswami, Umans and Vadhan gave an explicit construction of a lossless condenser based on Parvaresh-Vardy codes. This lossless condenser is a basic building block in many constructions, and, in particular, is behind the state of the art extractor constructions.

We give an alternative construction that is based on Multiplicity codes. While the bottom-line result is similar to the GUV result, the analysis is very different. In GUV (and Parvaresh-Vardy codes) the polynomial ring is closed to a finite field, and every polynomial is associated with related elements in the finite field. In our construction a polynomial from the polynomial ring is associated with its iterated derivatives. Our analysis boils down to solving a differential equation over a finite field, and uses previous techniques, introduced by Kopparty (in [9]) for the list-decoding setting. We also observe that these (and more general) questions were studied in differential algebra, and we use the terminology and result developed there.

We believe these techniques have the potential of getting better constructions and solving the current bottlenecks in the area.

1 Introduction

A condenser is a probabilistic mapping from a large universe \( \{0,1\}^n \) to a smaller universe \( \{0,1\}^m \) that preserves the entropy of not too large sets. More formally, \( C : \{0,1\}^n \times [D] \rightarrow \{0,1\}^m \) is a \((k_1, k_2, \epsilon)\) condenser, if for every distribution \( X \) on \( \{0,1\}^n \) with \( k_1 \) min-entropy, the output distribution \( C(X, U_D) \) is \( \epsilon \)-close to having \( k_2 \) min-entropy (see Definition 6 for a formal definition).

Ideally, we would like to explicitly build a condenser for any \( n, k_1 < n \), and \( \epsilon = \epsilon(n) > 0 \) and have \( D \) as small as possible, \( k_2 \) as close as possible to \( k_1 + \log(D) \), and have \( k_2 \) as close as possible to \( m \). Let us call \( d = \log(D) \) the seed length of \( C \), it measures the amount of randomness the probabilistic construction uses, and clearly the smaller the better. Similarly, let us call \( k_1 + d - k_2 \) the entropy loss of \( C \). The entropy loss measures the difference between the amount of entropy in the system \( (k_1 + d) \) and the amount of entropy we preserve \( (k_2) \), and we want it small. Finally, let us call \( m - k_2 \) the entropy gap of \( C \). The entropy gap measures how dense the output distribution \( C(X, U_D) \) is in its ambient space \( \{0,1\}^m \), and the smaller the better.

Thus, in this terminology, given \( n, k_1 \) and \( \epsilon \) we would like to find an explicit construction simultaneously minimizing the seed length, entropy loss and entropy gap of the condenser.
An important special case is when the entropy gap \( m - k_2 \) is 0, and then \( C \) is called a \((k_1, \epsilon)\) extractor. Non-explicitly, there are extractors (and so the entropy gap zero) with entropy loss \( 2 \log(\frac{1}{\epsilon}) + O(1) \) and seed length \( \log(n - k_1) + 2 \log(\frac{1}{\epsilon}) + O(1) \), and each one of these bounds is tight (even individually) [13].

Dodis et al. [3] observe that if we allow some entropy gap (and in particular even if it is only a constant) then non-explicitly the entropy loss dramatically drops to \( O(\log \log(\frac{1}{\epsilon})) \) and the seed length to \( \log(n - k) + 1 \cdot \log(\frac{1}{\epsilon}) + O(1) \). With larger entropy gaps, the entropy loss continues to drop until it basically turns into zero, and then we get a lossless condenser. For the dependence of the entropy loss on the entropy gap see [3] (and also [1]).

The GUV lossless condenser [7] has logarithmic seed length and constant fraction entropy gap. Specifically,

**Theorem 1.** (The GUV condenser)\cite{7}, Theorem 1.7 For every \( n \in \mathbb{N}, k_{\text{max}} \leq n, \epsilon > 0, \) and \( 0 < \alpha \leq 1 \), there exists an \( m \leq 2d + (1 + \alpha)k_{\text{max}} \) and an explicit function

\[
C : \{0, 1\}^n \times \{0, 1\}^d \rightarrow \{0, 1\}^m
\]

with \( d = (1 + 1/\alpha) \cdot (\log n + \log k_{\text{max}} + \log 1/\epsilon) + O(1) \) such that for all \( k \leq k_{\text{max}} \), \( C \) is an \((n, k) \rightarrow \epsilon (m, k + d) \) (lossless) condenser.

The GUV condenser has found numerous applications (as can be easily seen by looking at the hundreds of papers that cite it). In particular, GUV present an extractor construction by first applying the GUV lossless condenser, and then an extractor construction specifically designed for high min-entropy sources (see [7, Section 4]). Roughly speaking, this extractor construction inherits its entropy loss from the entropy gap of the lossless condenser. As a result, the extractor construction presented in [7] has linear entropy loss.

The problem of constructing explicit extractors with short seed length and small entropy loss is widely open and there has been only modest improvement over the extractor of [7] that has linear entropy loss. Specifically, [4] construct explicit extractors with the slightly sub-linear entropy loss \( \frac{k}{\log \log(k)} \). Their construction uses improved mergers that are obtained using the polynomial method with multiplicities. In another work, [15] modify the GUV condenser construction and using again the multiplicity method of [4] together with other ideas, give a condenser with small entropy loss and the slightly sub-linear entropy gap \( \frac{m}{\log \log(n)} \). This condenser implies an explicit extractor with a short seed and the same slightly sub-linear entropy loss. Constructing an extractor with a short seed and a better entropy loss is still a major open problem.

In this paper we give another explicit construction of a GUV like lossless condenser. While we do not improve the parameters, our construction uses a different analysis that we believe has the potential to substantially improve current state of the art results. Specifically, we prove:

**Theorem 2.** (Our condenser) For every \( n \in \mathbb{N}, k_{\text{max}} \leq n, \epsilon > 0, \) and \( \frac{16 \log 2}{\sqrt{k_{\text{max}}}} \leq \alpha \leq 1 \), there is an \( m \leq d + (1 + \alpha)k_{\text{max}} \) and an explicit function

\[
C : \{0, 1\}^n \times \{0, 1\}^d \rightarrow \{0, 1\}^m
\]

with \( d = (1 + 1/\alpha) \cdot (\log n + \log k_{\text{max}} + \log 1/\epsilon) + O(1) \) such that for all \( k \leq k_{\text{max}} \), \( C \) is an \((n, k) \rightarrow \epsilon (m, k + d) \) (lossless) condenser.

In a similar fashion to [7], our condenser follows from a new construction of an unbalanced bipartite expander graph.
Fix a field $\mathbb{F}_q$, $n, s \in \mathbb{N}$ such that $15 \leq s + 2 \leq n \leq \text{char}(\mathbb{F}_q)$, there exists an explicit graph $\Gamma : \mathbb{F}_q^n \times \mathbb{F}_q \rightarrow \mathbb{F}_q^{s+2}$, which is a $(K, A)$ expander for every $K > 0$ with

$$A = q - \frac{n(s + 2)}{2} \cdot (qK)^{\frac{s}{s+2}}.$$  \hspace{1cm} (1)

In [7] there is a similar expression with $A = q - (n - 1)(s + 1)(K^{\frac{s}{s+2}} - 1)$.

While the bound on $m$ in Theorem 2 is slightly better than the one in Theorem 1, the former has more restrictions on $\alpha$ then the latter. In any case, those two differences are minor, and as stated before, the main contribution of Theorem 2 is the method used to prove it, which is very different then the one used in [7], as we next explain.

1.1 Our construction and the GUV construction

Both our construction and the GUV construction have the following structure. The input that we want to condense is interpreted as a degree $n - 1$ univariate polynomial over $\mathbb{F}_q$, i.e., an element $f$ from $\mathbb{F}_q^n[X]$. Given the output length $s + 2 \in \mathbb{N}$ (with $s + 2 < n$) both constructions associate $f$ with $s + 1$ different polynomials $f_0, \ldots, f_s$ where $f_i \in \mathbb{F}_q^n[X]$. In GUV the association is done as follows:

1. First, put a field structure on $\mathbb{F}_q^n[X]$ and fix $h \in \mathbb{N}$, that way $f^{h i}$ (where multiplication and powering is in the field) can also be interpreted as a degree less than $n$ polynomial.
2. Define $f_i = f^{h i}$.

For example, one may choose a degree $n$ irreducible polynomial $E \in \mathbb{F}_q[X]$ and define the field $\mathbb{F} = \mathbb{F}_q[X] \mod E$. Then, the condenser construction is as follows:

**The condenser $C$**

**Parameters:** Fix a field $\mathbb{F}_q$, $n, s \in \mathbb{N}$, $n, s \geq 1$. Identify the elements of $\mathbb{F}_q^n$ with univariate polynomials of degree less than $n$.

**Construction:** Define $C : \mathbb{F}_q^n \times \mathbb{F}_q \rightarrow \mathbb{F}_q^{s+2}$ by:

$$C(f, y) = (y, f_0(y), f_1(y), \ldots, f_s(y)).$$ \hspace{1cm} (2)

Our construction has the same structure, but our choice of the associated functions $f_0, \ldots, f_s$ is different. Instead of choosing $f_0, \ldots, f_s$ as in GUV, we choose

$$f_i = f^{(i)},$$

i.e., $f^{(i)}$ is the $i$’th iterated derivative of $f$ in $\mathbb{F}_q[X]$.

To see why our construction is natural, let us look at it from a coding theory perspective. We can associate a function $C : V \times \Sigma \rightarrow \Sigma$ with $V$ a linear code of length $D$ and alphabet $\Sigma$, where for every $v \in V$ we have the codeword

$$(c(v)_1, \ldots, c(v)_D) \in \Sigma^D$$

where $c(v)_i = C(v, i)$. Using this translation, the GUV construction exactly corresponds to the PV code [12] and our construction exactly corresponds to Multiplicity codes [10, 8].

PV codes and Multiplicity codes are among the few explicit constructions of ECC with close to optimal list-decoding capacity. In the list-decoding problem our goal is to find a construction such that for every given word $(w_1, \ldots, w_D) \in \Sigma^D$ there are few $v \in V$ such that $c(v)$ is close to $w$. In the condenser construction problem we wish to solve a problem
similar to the list-recoverability problem, our input is a large subset $W \subseteq \Sigma$, and the output should be the (hopefully few) $v \in V$ such that $c(v) \in W$ for every $i \in [D]$ (or the variant where $c(v) \in W$ for most $i \in [D]$). Indeed, GUV write that the known connection between codes and extractors (pointed out, e.g., in [17]) and the fact that PV codes have list-decoding close to capacity motivated them to explore whether PV codes give condensers with good list-recoverability.

Looking at it from this perspective, in this paper we ask whether Multiplicity codes, which are known to have list-decoding close to capacity, also have good list-recoverability and hence give good condensers. In Theorems 2 and 3 we show that this is indeed the case.

Another code which has close to optimal list-decoding capacity is the Folded Reed-Solomon code defined in [6]. Consequently, the condenser it produces has been analyzed in [7, Section 6], and achieved worse parameters than the PV based condenser. Interestingly, the parameters are also worse than the ones achieved by our Multiplicity condenser, making this the first time, to the best of our knowledge, that a construction based on Multiplicity codes achieves better results than one based on FRS codes.

While our construction and the GUV construction are similar in structure, they are very different in implementation. In GUV the ring of polynomials $\mathbb{F}_q^n[X]$ is “lifted” to a finite field, and the associated functions $f_i$ are chosen so that they lie on a curve, specifically, over the extension field $\mathbb{F}$, all the functions $f_i$ are just polynomials in one common variable. The challenge is proving that if $Q(y,f_0,\ldots,f_m)$ is a non-zero polynomial in the polynomial ring, then $Q$ composed with the curve is a non-zero, univariate polynomial over the extension field $\mathbb{F}$. In general, proving that a non-zero polynomial composed with a given curve remains non-zero is a non-trivial challenge, and GUV solve it with a specific trick, that works, but gives constant entropy gap.

In contrast, our construction does not lift to an extension field. Instead the associated functions are just the derivatives of the given input. Thus, we completely avoid the question of proving that a non-zero polynomial composed with a curve remains non-zero, and, instead, we are left with a question similar to interpolation from derivatives. This leads to a widely different analysis as we explain next. We hope that further extensions of it might lead to constructions better than the current state of the art.

1.2 The proof technique

We give a proof sketch of Theorem 3 (the expanding graph). It is enough to prove that for every $W \subseteq \mathbb{F}_{q+2}^s$ of size at most $AK - 1$ we have $|\text{LIST}(W)| < K$. Fix a set $W \subseteq \mathbb{F}_{q+2}^s$ of size $AK - 1$. Our goal is to bound the number of degree $n - 1$ polynomials $f$ such that $\Gamma(f) \subseteq W$.

Our starting point is to find a non-zero, low-degree, multi-variate polynomial $Q(X,Y_0,\ldots,Y_s)$ such that $Q(w) = 0$ for every $w \in W$. This step is identical to the first step in the proof of GUV. The total degree of $Q$ is $O(|W|^{1/(s+2)}s)$. It is a standard observation that for every $f$ with $\Gamma(f) \subseteq W$ it must be that

$$Q \circ df = Q(x,f(x),f'(x),\ldots,f^{(s)}(x))$$

is the zero polynomial, i.e., $f$ solves the differential equation $Q$. The challenge now is to bound that number of functions $f$ such that $\Gamma(f) \subseteq W$.

To bound the number of degree $n - 1$ polynomials such that $\Gamma(f) \subseteq W$ we adapt the list-decoding algorithm of [9] to the list-recovery setting (much the same as GUV adapt the [12] list-decoding algorithm to the list-recovery setting). The main lemma Kopparty uses is
that given \((y, w_0, \ldots, w_s) \in \mathbb{F}_q \times \mathbb{F}_q^{s+1}\), there is usually at most one degree \(n - 1\) polynomial \(f\) such that:

- The first \(s\) derivatives of \(f\) at \(y\) agree with \(w_0, \ldots, w_s\), i.e., \(f^{(i)}(y) = w_i\), for \(i = 1, \ldots, s\), and,
- \(Q \circ df\) is the zero polynomial.

Formally, this is true whenever the Separant of the equation, \(\frac{\partial Q}{\partial Y}\), is non-singular at \(w\), i.e.,
\[
\frac{\partial Q}{\partial Y}(y, w_0, \ldots, w_s) \neq 0.
\]

Kopparty proves this lemma using Hensel lifting. We rephrase the proof using differential algebra terminology and intuition from [14]. We believe our proof is simpler, and also more amenable to generalizations. Furthermore, this theory was generalized in [11, 5], where generalized Separants were introduced, and we believe these generalizations might be useful for future improvements of the analysis.

Going back to the list-recovery problem, and following the list-decoding algorithm from [9], let us denote by \(W_1\) the set of all \(w \in W\) such that \(\frac{\partial Q}{\partial Y}(w) \neq 0\). We see that for every \(f\) such that \(\text{supp}(f) \subseteq W_0 = W \ setminus W_1\), we can recover \(f\) by going over all \(w \in W_1\), and for each such \(w\) output the unique suitable degree \(n - 1\) polynomial, given by the above main lemma.

We are then left with the task of outputting all the degree \(n - 1\) polynomials such that \(\text{supp}(f) \subseteq W_0 = W \ setminus W_1\). We notice that each of these polynomials solve the lower degree differential equation \(\frac{\partial Q}{\partial Y}(x, f(x), \ldots, f^{(s)}) = 0\). Reiterating the process we get a new list of solution. As each time we get a lower degree differential equation, we can iterate the process at most \(\deg(Q)\) times. Doing the calculation more carefully (as is done in [9]) saves even this loss, and, furthermore, shows expansion by a factor of about \(q - s \cdot \sqrt{|W|}\). We explain the thin details in Section 4.

## 2 Preliminaries

We use the following notation:

\[
(n)_t = n \cdot (n - 1) \cdot \ldots \cdot (n - t + 1) = \frac{n!}{(n - t)!},
\]

where for \(t = 0\), \((n)_0 = 1\). Thus, \((n)_t = t! \binom{n}{t}\).

Also, for \(J = (j_1, \ldots, j_m)\) and \(I = (i_1, \ldots, i_m)\) we define

\[
(J)_I = \prod_{\ell=1}^{m} (j_{\ell})_{i_{\ell}},
\]

\[
\binom{J}{I} = \prod_{\ell=1}^{m} \binom{j_{\ell}}{i_{\ell}}, \text{ and,}
\]

\[
I! = \prod_{\ell=1}^{m} i_{\ell}!.
\]

Thus, \((J)_I = I! \binom{J}{I}\). Finally, \(J - I = (j_1 - i_1, \ldots, j_m - i_m)\).

### 2.1 Multi-variate derivatives

Let \(R = \mathbb{F}[X_1, \ldots, X_m]\) be the ring of polynomials in \(m\) variables over \(\mathbb{F}\). For \(I = (i_1, \ldots, i_m)\) with \(i_1, \ldots, i_m \in \mathbb{N}\) we define the partial derivative in direction \(I\) as the linear operator on \(R\)
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defined by \( \frac{\partial X^J}{\partial t} = (J)_t \cdot X^{J-t} \). We denote

\[
Q^{(i)}(X) = \frac{\partial Q}{\partial I}(X).
\]

The order of \( \Gamma \) is \( w(\Gamma) = i_1 + \ldots + i_m \). Notice that for uni-variate polynomials \( Q(X) \), \( Q^{(i)}(X) \) coincides with the \( i \)th iterated derivative.

Let \( w = (w_1, \ldots, w_m) \) where \( w_i \in \mathbb{N} \). The \( w \)-weighted degree of a monomial \( X^J = X_1^{j_1} \cdots X_m^{j_m} \) is \( \sum_{i=1}^{m} w_i \cdot j_i \). The \( w \)-weighted degree of \( Q \), denoted \( \deg_w(Q) \), is the largest \( w \)-weighted degree of a monomial in \( Q \). We let \( |w| \) denote \( \sum w_i \), \( \Pi(w) = \Pi w_i \), and \( M_{w,t} \) the number of monomials \( X^J \) with \( w \)-weighted degree at most \( t \). Beged-Dov gave upper and lower bounds on \( M_{w,t} \):

\[ \frac{t^m}{m! \cdot \Pi(w)} \leq M_{w,t} \leq \frac{(t + |w|)^m}{m! \cdot \Pi(w)} \]

2.2 Condensers

In this subsection let \( C : \{0,1\}^{n} \times \{0,1\}^{d} \rightarrow \{0,1\}^{m} \).

\begin{definition}
We say \( C \) is a \((K,A)\) expander if for every \( S \subseteq \{0,1\}^{n} \) of cardinality \( K \) the set

\[ \Gamma(S) = \bigcup_{s \in S, \gamma \in \{0,1\}^{d}} C(s, \gamma) \]

has cardinality at least \( K \cdot A \).
\end{definition}

We next define a condenser:

\begin{definition}
We say \( C \) is an \((n,k) \rightarrow_{\epsilon} (m,k')\) condenser if for all distributions \( X \) with min-entropy at least \( k \), the distribution \( C(X, U_d) \) is \( \epsilon \)-close to a distribution with min-entropy at least \( k' \). The condenser is explicit if \( C \) can be computed in time \( \text{poly}(n, \frac{1}{\epsilon}) \).
\end{definition}

To prove that a function is a condenser or an expander, we use the “list-decoding” approach described in [7]. For \( C : \{0,1\}^{n} \times \{0,1\}^{d} \rightarrow \{0,1\}^{m} \) and \( T \subseteq \{0,1\}^{m} \) define:

\[
\text{LIST}(T) = \{ x : \Gamma(x) \subseteq T \}
\]

\[
\text{LIST}(T, \epsilon) = \left\{ \frac{\Gamma(x,y)}{y} : \frac{\Gamma(x, y)}{y} \in T \right\}
\]

\begin{lemma}
[7, Lemma 3.2] \( C \) is a \((K,A)\) expander iff for every set \( T \subseteq \{0,1\}^{m} \) of cardinality at most \( AK - 1 \), \( \text{LIST}(T) \) has cardinality at most \( K - 1 \).
\end{lemma}

And for condensers:

\begin{lemma}
[16, Theorem 8.1],[7, Lemma 5.4] Let \( C : \{0,1\}^{n} \times \{0,1\}^{d} \rightarrow \{0,1\}^{m} \) be a function.

- If \( C \) is a \((K, (1-\epsilon)^2d)\) expander, then \( C \) is a \((n,k) \rightarrow_{\epsilon} (m,k+d)\) condenser, i.e., it is a lossless condenser with error \( \epsilon \).
- If for all \( T \subseteq \{0,1\}^{m} \) of size at most \( L \) the set \( \text{LIST}(T, \epsilon) \) has cardinality at most \( H \), then \( C \) is a \((n, \log(\frac{H}{\epsilon})) \rightarrow_{2\epsilon} (m, \log(\frac{L}{\epsilon}) - 1)\) condenser.
\end{lemma}
3 The Separant

Let \( Q \in \mathbb{F}_q[X,Y_0,\ldots,Y_s] \). When we think of \( Q \) as a differential equation, we look for all (low-degree) polynomials \( f \in \mathbb{F}_q[X] \) such that

\[
Q(X,f(X),f^{(1)}(X),\ldots,f^{(s)}(X)) = 0 \in \mathbb{F}_q[X].
\]

Let us define

\[
\overline{df} = (X,f(X),f^{(1)}(X),\ldots,f^{(s)}(X),\ldots)
\]

Notice that if \( f \in \mathbb{F}_q^n[X] \), then \( f^{(i)}(X) \) is identically zero for all \( i \geq n \). Let us also think of \( Q \) as a polynomial \( Q \in \mathbb{F}_q[X,Y_0,\ldots,Y_s,\ldots,Y_{n\ldots}] \) that depends only on \( X \) and \( Y_0,\ldots,Y_s \).

In this notation \( f \) solves the differential equation \( Q \) iff \( Q \circ \overline{df} = 0 \in \mathbb{F}_q[X] \).

A differential equation \( Q \) can be itself derived. While formally \( Q \) depends on \( X \) and \( Y_0,\ldots,Y_s,\ldots \), we think of \( Y_0 \) as a function depending on \( X,Y_0 = f(X) \) and of \( Y_{i+1} \) as \( \frac{\partial Y_i}{\partial X} \). This motivates the following definition:

\[\text{Definition 9.} \quad \text{Let } Q \in \mathbb{F}_q[X,Y_0,\ldots,Y_s], \text{ define the infinite sequence of polynomials } Q^{(0)},Q^{(1)},\ldots \text{ where } Q^{(k)} \in \mathbb{F}_q[X,Y_0,\ldots,Y_{k+s}] \text{ is defined by:}\]

\[
Q^{(0)} = Q
\]

\[
Q^{(k+1)} = \frac{\partial Q^{(k)}}{\partial X} + \sum_{i=0}^{k+s} \frac{\partial Q^{(k)}}{\partial Y_i} \cdot Y_{i+1}.
\]

The motivation behind this definition is apparent given:

\[\text{Lemma 10.} \quad \text{For every } f \in \mathbb{F}_q[X] \text{ and } \ell \geq 0 \]

\[
(Q \circ \overline{df})^{(\ell)} = Q^{(\ell)} \circ \overline{df}.
\]

\[\text{Proof.} \quad \text{By induction. The case } \ell = 0 \text{ is immediate. Assume for } \ell \text{ and let us prove for } \ell + 1. \]

Using the chain rule:

\[
(Q \circ \overline{df})^{(\ell+1)} = ((Q \circ \overline{df})^{(\ell)})' = (Q^{(\ell)} \circ \overline{df})'
\]

\[
= \frac{\partial Q^{(\ell)}}{\partial X} \circ \overline{df} + \sum_{i=0}^{\ell+s} \frac{\partial Q^{(\ell)}}{\partial Y_i} \circ \overline{df} \cdot \frac{\partial f^{(i)}}{\partial X}
\]

\[
= \frac{\partial Q^{(\ell)}}{\partial X} \circ \overline{df} + \sum_{i=0}^{\ell+s} \frac{\partial Q^{(\ell)}}{\partial Y_i} \circ \overline{df} \cdot f^{(i+1)}
\]

\[
= (\frac{\partial Q^{(\ell)}}{\partial X} + \sum_{i=0}^{\ell+s} \frac{\partial Q^{(\ell)}}{\partial Y_i} \cdot Y_{i+1}) \circ \overline{df}
\]

\[
= Q^{(\ell+1)} \circ \overline{df},
\]

where the first equality is because we use iterated derivations, the second is induction, the third is the chain rule (and notice that \( Q^{(\ell)} \) depends on \( X,Y_0,\ldots,Y_{\ell+s} \)).

We call \( Q^{(\ell)} \) the \( \ell \)-th derivative of \( Q \). This operation comes from differential algebra [14]. As its name suggests, this operator has some properties similar to regular derivative

\[\text{Claim 11.} \quad [14]\]
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1. (linearity) For every $Q, P \in \mathbb{F}_q[X, Y_0, \ldots], \lambda, \mu \in \mathbb{F}_q, \ell \geq 0$
   \[ (\lambda Q + \mu P)^{(\ell)} = \lambda Q^{(\ell)} + \mu P^{(\ell)} \]

2. (Leibniz product rule) For every $Q, P \in \mathbb{F}_q[X, Y_0, \ldots]$
   \[ (P \cdot Q)^{(1)} = P^{(1)} \cdot Q + P \cdot Q^{(1)} \]

3. (repeated derivation) For every $Q \in \mathbb{F}_q[X, Y_0, \ldots], \ell_1, \ell_2 \geq 0$
   \[ (Q^{(\ell_1)})^{(\ell_2)} = Q^{(\ell_1+\ell_2)} \]

▶ Claim 12. Let $Q \in \mathbb{F}_q[X, Y_0, \ldots, Y_s]$ and $\ell \in \mathbb{N}$.
   = $\deg_{(0,1,1,\ldots)} Q^{(\ell)} = \deg_{(0,1,1,\ldots)} Q$, and,
   = $\deg_{(0,1,2,3,\ldots)} Q^{(\ell)} \leq \ell$. I.e., if we give $X, Y_0, \ldots, Y_s$ weight 0, and $Y_{s+j}$ weight $j$, then the $\ell$th derivative degree is at most $\ell$.

Proof. For the first item notice that $\frac{\partial Q^{(\ell)}}{\partial X}$ is either zero or does not change the degree in $Y_0, \ldots$. Also, the effect of $\frac{\partial Q}{\partial Y_s} \cdot Y_{i+1}$ is to reduce the degree in $Y_i$ by one and increase the degree in $Y_{i+1}$ by one.

For the second item, we prove by induction. The case $\ell = 0$ is immediate. For the induction step, $\frac{\partial Q^{(\ell)}}{\partial X}$ and $\frac{\partial Q^{(\ell)}}{\partial Y_s} \cdot Y_{i+1}$ for $i < s$, are either zero or do not change the weighted degree, while $\frac{\partial Q^{(\ell)}}{\partial Y_s} \cdot Y_{i+1}$ for $i \geq s$ increase the weighted degree by one. ▶

One consequence of Claim 12 is that $Y_{s+\ell}$ appears with degree at most 1 in $Q^{(\ell)}$ and that the coefficient of $Y_{s+\ell}$ in $Q^{(\ell)}$ is a function of $X, Y_0, \ldots, Y_s$ alone. Indeed, we next prove the coefficient of $Y_{s+\ell}$ in $Q^{(\ell)}$ is $\frac{\partial Q}{\partial Y_s}$.

▶ Definition 13. (Separant) Let $Q \in \mathbb{F}[X, Y_0, \ldots, Y_s]$. The Separant of $Q$, denoted $S_Q$, is
   \[ S_Q = \frac{\partial Q}{\partial Y_s}. \]

A classical lemma from differential algebra (see [14, Page 30]) states that:

▶ Lemma 14. For every $\ell \geq 1$, $Q^{(\ell)} = S_Q \cdot Y_{s+\ell} + R_\ell$ where $R_\ell \in \mathbb{F}[X, Y_0, \ldots, Y_{s+\ell-1}]$ does not depend on $Y_{s+\ell}$.

Proof. By induction. For $\ell = 1$, the only way to get $Y_{s+1}$ in $Q^{(1)}$ is in the term $\frac{\partial Q}{\partial Y_s} \cdot Y_{s+1}$. Assume for $\ell$ and let us prove for $\ell + 1$. The only way to get $Y_{s+\ell+1}$ in $Q^{(\ell+1)}$ is by taking $\frac{\partial Q^{(\ell)}}{\partial Y_s}$. By induction, $Y_{s+\ell}$ only appears in $Q^{(\ell)}$ in the linear term $S_Q \cdot Y_{s+\ell}$. Thus, the only term involving $Y_{s+\ell+1}$ in $Q^{(\ell+1)}$ is $\frac{\partial(S_Q \cdot Y_{s+\ell})}{\partial Y_{s+\ell}} \cdot Y_{s+\ell+1} = S_Q \cdot Y_{s+\ell+1}$. ▶

▶ Lemma 15. Fix $Q \in \mathbb{F}_q[X, Y_0, \ldots, Y_s]$, $(\alpha, b) = (\alpha, b_0, \ldots, b_s) \in \mathbb{F}_q^{s+2}$ and $S_Q(\alpha, b) \neq 0$. Suppose $f \in \mathbb{F}_q[X]$ such that:
   = $f^{(i)}(\alpha) = b_i$, for $i = 0, \ldots, s$, and
   = $Q \circ df = 0$.

Then there are unique values $b_{s+1}, \ldots, b_n$ such that $f^{(i)}(\alpha) = b_i$. 

Proof. We prove by induction on $n$. The base case $n = s$ is clear. Assume for $n$ and let us prove for $n + 1$. By assumption we know there are unique values $b_{s+1}, \ldots, b_n$ such that $b_i = f^{(i)}(\alpha)$ for $i = s + 1, \ldots, n$. Our goal is to show there is a unique value possible for $f^{(n+1)}(\alpha)$.

We will use $Q^{(n-s+1)}$ and the fact that $Y_{n+1}$ appears linearly in it with coefficient $S_Q$, and that at $(\alpha, b)$, $S_Q(\alpha, b) \neq 0$. First we notice that

\[
Q^{(n-s+1)}(\alpha, b_0, \ldots, b_n, f^{(n+1)}(\alpha)) = Q^{(n-s+1)}(\alpha, f(\alpha), \ldots, f^{(n+1)}(\alpha))
\]

\[
= Q^{(n-s+1)} \circ \overline{df}(\alpha)
\]

\[
=(Q \circ \overline{df})^{(n-s+1)}(\alpha) = 0,
\]

where the first equality is by induction, the second by definition, the third using Lemma 10, and the last equality because we know $Q \circ \overline{df}$ is the zero polynomial in $\mathbb{F}_q[\alpha]$.

Next we recall that by Lemma 14

\[
Q^{(n-s+1)}(X, Y_0, \ldots, Y_{n+1}) = S_Q(X, Y_0, \ldots, Y_s) \cdot Y_{n+1} + R(X, Y_0, \ldots, Y_n),
\]

and therefore

\[
0 = Q^{(n-s+1)}(\alpha, b_0, \ldots, b_n, f^{(n+1)}(\alpha))
\]

\[
= S_Q(\alpha, b) \cdot f^{(n+1)}(\alpha) + R(\alpha, b_0, \ldots, b_n).
\]

Thus, $f^{(n+1)}(\alpha) = -\frac{R(\alpha, b_0, \ldots, b_n)}{S_Q(\alpha, b)}$ is uniquely determined. ▷

In words, this means the following. $f$ solves the differential equation if $Q \circ \overline{df} = 0$. We can think of the conditions $f^{(i)}(\alpha) = b_i$, for $i = 0, \ldots, s$, as $s + 1$ initial conditions on the Taylor expansion of $f$ at $\alpha$. In this terminology, Lemma 15 says that that if the separant $S_Q$ is non-zero at the point $(\alpha, b)$ then there can be at most one solution to the differential equation $Q$ with degree smaller than the characteristic, satisfying the initial conditions $(\alpha, b)$.

## 4 Reconstruction with the Polynomial Method

In this section we present a “de-condensing” procedure that given $\Gamma: \mathbb{F}_q^n \times \mathbb{F}_q \rightarrow \mathbb{F}_q^{s+2}$ and a set $W \subseteq \mathbb{F}_q^{s+2}$ outputs LIST($W$). Throughout this section we assume that $n \leq \text{char}(\mathbb{F}_q)$.

The de-condensing algorithm works as follows. Given $W$ we first find a low-degree polynomial $Q$ that vanishes over $W$, namely,

▶ Claim 16. There exists a non-zero polynomial $Q \in \mathbb{F}_q[X, Y_0, \ldots, Y_s]$ with

\[
\deg_{(1, n, \ldots, n-s)} Q \leq D = \left\lceil n \cdot \left( |W| \cdot (s+2) \right)^{1/s} \right\rceil
\]

that vanishes on $W$.

Proof. By Lemma 4 the number of monomials in $\mathbb{F}_q[X, Y_0, \ldots, Y_s]$ with $(1, n, n-1, \ldots, n-s)$-weighted degree at most $D$ is some value $F$ such that

\[
F \geq \frac{D^{s+2}}{(s+2)! \cdot \prod_{j=0}^{n-1} (n-j)} > |W|.
\]

To find a polynomial $Q$ that vanishes on $W$, we write a homogeneous linear system over $\mathbb{F}_q$ where the variables are the coefficients of the above monomials, and for every $w \in W$ we have a linear equation forcing that the polynomial vanishes on $w$. As the number of variables is larger than the number of constraints, there is a non-zero solution. ▷
It then follows that every \( f \in \mathbb{F}_q^{<n}[T] \) with \( \Gamma(f) \subseteq W \) satisfies the differential equation \( Q(x, f(x), \ldots, f^{(s)}(x)) = 0 \). Formally,

\[ \text{Claim 17. If } f \in \text{LIST}(W), \text{ and } q > D, \text{ then } \]

\[ R_f(T) = Q(T, f(T), \ldots, f^{(s)}(T)) \in \mathbb{F}_q[T] \]

is the zero polynomial.

**Proof.** As \( \deg(1, n, \ldots, n-s)(Q) \leq D \) and \( \deg(f^{(i)}) < n - i, R_f \) has degree at most \( D \). Also, for every \( \alpha \in \mathbb{F}_q \),

\[ R_f(\alpha) = Q(\alpha, f(\alpha), \ldots, f^{(s)}(\alpha)) = 0. \]

As \( q > D \) we must have \( R_f = 0 \) in \( \mathbb{F}_q[T] \). ▶

The main challenge is proving the number of low-degree solutions to the differential equation \( Q \) with starting conditions \( W \) is small, and designing an algorithm finding all such solutions. For that we define algorithm \textit{Solve}. The input to the algorithm is a polynomial \( \dot{Q} \in \mathbb{F}_q[X, Y_0, \ldots, Y_s] \) and \( \dot{W} \subseteq \mathbb{F}_q^{s+2} \). The output contains all polynomials \( f \in \mathbb{F}_q^{<n}[X] \) such that \( \Gamma(f) \subseteq \dot{W} \) and \( \dot{Q} \circ \frac{df}{dT} = 0 \). The algorithm works as follows:

\begin{algorithm}
\caption{Solve(\( \dot{Q}, \dot{W} \))}
\begin{algorithmic}[1]
\STATE If \( \dot{Q} \) does not depend on \( Y_0, \ldots, Y_s \) return \( \emptyset \).
\STATE Let \( s^* \) be the largest \( j \in \{0, \ldots, s\} \) for which \( \dot{Q} \) depends on \( Y_j \).
\STATE Set \( L_1 \leftarrow \emptyset \) and \( \dot{W}_1 \leftarrow \left\{ w \in \dot{W} \mid \frac{\partial \dot{Q}}{\partial Y_{s^*}} (w) \neq 0 \right\} \).
\FOR{\( w = (\alpha, w_0, \ldots, w_s) \in \dot{W}_1 \)}
\STATE Assuming there exits some polynomial \( g \in \mathbb{F}_q[X] \) such that \( \dot{Q} \circ \frac{dg}{dT} = 0 \in \mathbb{F}_q[X] \) and \( g^{(i)}(\alpha) = w_i \) for all \( 0 \leq i \leq s \), find the unique values \( w_{s+1}, \ldots, w_{n-1} \) such that \( g^{(i)}(\alpha) = w_i \) for all \( 0 \leq i < n \). Such a unique solution exits by Lemma 15.
\STATE Define \( f(x) = \sum_{i=0}^{n-1} \frac{w_i}{i!} (x - \alpha)^i \).
\STATE If \( \Gamma(f) \subseteq \dot{W} \) add \( f \) to \( L_1 \).
\STATE Set \( \dot{W}_0 \leftarrow \left\{ w \in \dot{W} \mid \frac{\partial \dot{Q}}{\partial Y_{s^*}} (w) = 0 \right\} \).
\STATE \( L_0 \leftarrow \text{Solve}(\frac{\partial \dot{Q}}{\partial Y_{s^*}}, \dot{W}_0) \)
\STATE return \( L_0 \cup L_1 \)
\end{algorithmic}
\end{algorithm}

With that the de-condensing algorithm is:
Algorithm 2 Decondensing

**Input:** Parameters \( q, s, n \), the condenser \( \Gamma : \mathbb{F}_q^n \times \mathbb{F}_q \rightarrow \mathbb{F}_q^{s+2} \), and a set \( W \subseteq \mathbb{F}_q^{s+2} \)

**Output:** All \( f \in \mathbb{F}_q^{s+1}[X] \) such that \( \Gamma(f) \subseteq W \)

1. Set \( D \leftarrow n \cdot \left[ \left| W \right| \cdot (s + 2) \right]^{1/2} \)
2. Construct a non-zero polynomial \( Q \in \mathbb{F}_q[X, Y_0, \ldots, Y_s] \) with
   \[
   \deg(1, n, \ldots, n-s) Q \leq D
   \]
   that vanishes on \( W \).
3. **return** \( \text{Solve}(Q, W) \)

### 4.1 Analysis of Solve

**Lemma 18.** (Correctness of Solve) Fix a non-zero polynomial \( Q \in \mathbb{F}_q[X, Y_0, \ldots, Y_s] \) such that \( \deg(1, n, n-s) Q < q \), and \( W \subseteq \mathbb{F}_q^{s+2} \). Every \( f \in \mathbb{F}_q^{s+1}[T] \) for which \( \Gamma(f) \subseteq W \) and \( Q(x, f(x), \ldots, f^{(s)}(x)) = 0 \) appears in the output of Solve\((Q, W)\).

**Proof.** The proof is by induction on the degree of \( Q \) as a polynomial in \( Y_0, \ldots, Y_s \), i.e., \( \deg(0, 1, \ldots, 1)(Q) \). In the base case \( Q \) depends only on \( X \), thus \( Q = Q(X) \). As \( Q \neq 0 \), there are no solutions to \( Q(T, f(T), \ldots, f^{(s)}(T)) = 0 \) and \( \mathcal{L} = \emptyset \) is the correct output.

Now let \( f(T) \in \mathbb{F}_q^{s+1}[T] \) such that \( \Gamma(f) \subseteq W \) and \( Q(T, f(T), \ldots, f^{(s)}(T)) = 0 \). We have two cases:

1. \( \frac{\partial Q}{\partial Y_s}(T, f(T), \ldots, f^{(s)}(T)) \neq 0 \). Note that
   \[
   \deg \left( \frac{\partial Q}{\partial Y_s}(T, f(T), \ldots, f^{(s)}(T)) \right) \leq \deg(1, n, \ldots, n-s) \left( \frac{\partial Q}{\partial Y_s}(X, Y_0, \ldots, Y_s) \right)
   \]
   \[
   \leq \deg(1, n, \ldots, n-s) (Q(X, Y_0, \ldots, Y_s)) < q.
   \]
   Therefore there must be some \( \alpha \in \mathbb{F}_q \) for which
   \[
   \frac{\partial Q}{\partial Y_s}(\alpha, f(\alpha), \ldots, f^{(s)}(\alpha)) \neq 0.
   \]
   As \((\alpha, f(\alpha), \ldots, f^{(s)}(\alpha)) \) \( \in \Gamma(f) \subseteq W \), in the for loop we iterate over this vector and therefore in line 5 we find the unique solution of the ODE with these initial conditions, and because of the uniqueness this solution must be \( f \). As \( \Gamma(f) \subseteq W \) we add it to the list \( \mathcal{L} \) in line 7.

2. \( \frac{\partial Q}{\partial Y_s}(T, f(T), \ldots, f^{(s)}(T)) = 0 \). We notice that in this case \( \Gamma(f) \subseteq W_0 \), as for every \( \alpha \in \mathbb{F}_q \) we have \( \frac{\partial Q}{\partial Y_s}(\alpha, f(\alpha), \ldots, f^{(s)}(\alpha)) = 0 \). Also \( \deg(0, 1, \ldots, 1) \left( \frac{\partial Q}{\partial Y_s} \right) < \deg(0, 1, \ldots, 1)(Q) \), hence by induction \( f \in \mathcal{L}_0 \).

**Lemma 19.** (List size of Solve) For every non-zero \( Q \in \mathbb{F}_q[X, Y_0, \ldots, Y_s] \) with \( \deg(1, n, n-1, \ldots, n-s)(Q) \leq D < q \) and every \( W \subseteq \mathbb{F}_q^{s+2} \), the size of the output of Solve\((Q, W)\) is at most \( \frac{|W|}{q-D} \).

**Proof.** We prove by induction on the \((0, 1, \ldots, 1)\)-degree of \( Q \). If \( \deg(0, 1, \ldots, 1)(Q) \) is zero, the list is empty, the list size is zero and the claim holds. We next prove the induction step.
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For every \( w = (\alpha, w_0, \ldots, w_s) \in W_1 \), there exists a unique \( f \) that may be joined to the list. Furthermore, since \( w \in W_1 \) we have that:

\[
\frac{\partial Q}{\partial Y_{s^*}}(\alpha, f(\alpha), \ldots, f(s)(\alpha)) = \frac{\partial Q}{\partial Y_{s^*}}(\alpha, w_0, \ldots, w_s) \neq 0,
\]

thus \( \frac{\partial Q}{\partial Y_{s^*}}(T, f(T), \ldots, f(s)(T)) \neq 0 \), and its degree is at most \( D \), meaning that it equals 0 for at most \( D \) values of \( T \), hence it is non-zero for at least \( q - D \) values of \( T \in \mathbb{F}_q \). Also, if \( f \) appears in the list then \( \Gamma(f) \subseteq W \). Hence, each of those \( q - D \) values lies in \( W \) (and therefore in \( W_1 \)) and reconstructs \( f \). We conclude that \( f \) is reconstructed from at least \( q - D \) different points in \( W_1 \), thus \( |\mathcal{L}_1| \leq \frac{|W_1|}{q - D} \).

We remain with the list size of \( \mathcal{L}_0 \) which is obtained from \( \text{Solve}(\frac{\partial Q}{\partial Y_{s^*}}, W_0) \). Since \( \deg_{(0,1, \ldots, 1)}(\frac{\partial Q}{\partial Y_{s^*}}) < \deg_{(0,1, \ldots, 1)}(Q) \), and the \((1, n, \ldots, n - s)\)-weighted degree of \( \frac{\partial Q}{\partial Y_{s^*}} \) is at most \( D \), we know by induction that \( |\mathcal{L}_0| \leq \frac{|W_0|}{q - D} \). Altogether, \( |\mathcal{L}| \leq \frac{|W_1|}{q - D} + \frac{|W_0|}{q - D} = \frac{|W|}{q - D} \). ▶

4.2 Putting it together

**Proof.** (of Theorem 3) By Lemma 7 it is enough to prove that for every \( W \subseteq \mathbb{F}_q^{s+2} \) of size at most \( AK - 1 \) we have \( |\text{LIST}(W)| < K \). Fix a set \( W \subseteq \mathbb{F}_q^{s+2} \) of size \( AK - 1 < qK \). Let \( Q \) be as in Claim 16, with

\[
D = \left[ n \cdot (qK \cdot (s+2)!)^{\frac{1}{s+2}} \right] \leq n \cdot (qK)^{\frac{1}{s+2}} \cdot (((s + 2)!)^{\frac{1}{s+2}} + 1)
\]

\[
\leq \frac{n(s + 2)}{2} \cdot (qK)^{\frac{1}{s+2}} = q - A
\]

Where the second to last inequality is due to the fact that \((k!)^{1/k} + 1 \leq \frac{k}{2} \) for every \( k \geq 15 \). Let \( \mathcal{L} \) be the output list of \( \text{Solve}(Q, W) \). Then,

\[
|\text{LIST}(W)| \leq |\mathcal{L}| \leq \frac{|W|}{q - D} \leq \frac{AK - 1}{q - D} < K,
\]

where the first inequality is by Lemma 18, the second by Lemma 19 and the last inequality by using the fact that \( A \leq q - D \). ▶

By choosing the parameters of in the same way as done in [7, Theorem 3.5] we get the following expander

**Theorem 20.** For every positive integers \( N, K_{\max} \leq N, \alpha > 0, \) and \( \frac{16 \log(\frac{14N}{\log K_{\max}})}{\sqrt{\log K_{\max}}} \leq \alpha \leq 1 \), there is an \( M \leq D \cdot K^{1+a}_{\max} \) and an explicit \((K_{\max}, (1 - \epsilon)D)\) expander \( \Gamma : [N] \times [D] \rightarrow [M] \) with degree \( D = O((\log N(\log K_{\max}))/\epsilon)^{1+1/\alpha} \).

For completeness we repeat the proof from [7].

**Proof.** Let \( n = \log N \) and \( k = \log K_{\max} \). Let \( h_0 = (2nk/\epsilon)^{1/\alpha} \), \( h = [h_0] \), and let \( q \) be a prime in the interval \((h^{1+\alpha}/2, h^{1+\alpha}]\).

Set \( s + 2 = [k/\log h] \), so that \( h^{s+1} \leq K_{\max} \leq h^{s+2} \). As \( 15 \leq s + 2 \leq n \leq q = \text{char}(\mathbb{F}_q) \), by Theorem 3, the graph \( \Gamma : \mathbb{F}_q^n \times \mathbb{F}_q \rightarrow \mathbb{F}_q^{s+2} \) is a \((\leq h^{s+2}, A)\) expander for \( A = q - \frac{n(s+2)}{2} \cdot (qK)^{\frac{1}{s+2}} \), because \( K_{\max} \leq h^{s+2} \), it is also a \((K_{\max}, A)\) expander.

Note that the number of left-vertices in \( \Gamma \) is \( q^n \geq N \), and the number of right-vertices is

\[
M = q^{s+2} \leq q \cdot h^{((1+\alpha)(s+1))} \leq q \cdot K^{1+\alpha}_{\max}
\]
The degree is
\[ D = q \leq h^{1+\alpha} \leq (h_0 + 1)^{1+\alpha} \]
\[ = O(h_0^{1+\alpha}) = O((nk/\epsilon)^{1+1/\alpha}) \]

Lastly, we consider the expansion factor, \( A = q - \frac{n(s+2)}{2} \cdot \left(\frac{qK}{h}\right)^{1/2} \geq q - \frac{nk\epsilon^{1/2} h_0^{1+\alpha}}{2} \), of the graph, first notice
\[ nkh \leq \epsilon h^{1+\alpha} \leq \epsilon q \]
where the first equality is due to the fact that \( nk/\epsilon \leq h^{\alpha/2} \). Secondly, we can convert our lower bound on \( \alpha \) to a lower bound on \( k \)
\[ k \geq \frac{256}{\alpha^2} \log \left(\frac{n}{\epsilon}\right) \]
and by using it we get
\[ s + 2 \geq \frac{k}{\log h} \geq \frac{256}{\alpha^2} \frac{\log^2 \left(\frac{n}{\epsilon}\right)}{\log h} \geq \frac{64}{\alpha^2} \frac{\log^2 (2nk)}{\log h} \]
\[ = \frac{16 \log^2 h_0}{\log h} \geq 4 \frac{\log^2 h}{\log h} = 4 \log h \geq (1 + \alpha) \log h \geq \log q \]
by combining the two inequalities
\[ \frac{nk\epsilon^{1/2} h_0^{1+\alpha}}{2} = nkh \cdot \frac{q^{1/(s+2)}}{2} \leq \epsilon q. \]
By substituting back to \( A \) we get \( A \geq (1 - \epsilon)q = (1 - \epsilon)D \), which concludes the proof.

Finally, Theorem 2 is an immediate consequence of Lemma 8 applied to Theorem 20.

References

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