General instructions:

1. The deadline for the exam is 17/02/19.
2. Submit your (typed) solution by mail to amnon@tau.ac.il.
3. Work must be done alone.
4. If you had to use an electronic source, state it explicitly within the relevant question.
1 \(k\)-wise independence (15 points)

Our goal in this question is to prove:

**Claim 1.** Let \(X \subseteq \{0,1\}^n\) be a \(k\)-wise independent distribution over \(\{0,1\}^n\), then \(|X| \geq \Omega \left( \frac{n^{k/2}}{2} \right)\)

1. Give an explicit construction that given a prime power \(n\) and \(k \leq n\) outputs an \(n \times k\) matrix over \(\mathbb{F}_n\) such that any \(k \times k\) minor is invertible over \(\mathbb{F}_n\).

2. Give an explicit construction of a pairwise independent distribution \(D\) over \(\mathbb{F}_2^{n-1}\) with \(|D| = n\).

3. Prove that \(D \subseteq \{0,1\}^n\) is a \(k\)-wise independent distribution if and only if \(\hat{D}(S) = 0\) for every \(S\) with \(0 < |S| \leq k\).

4. Let \(d = \frac{k^2}{2}\) and denote \(t = \sum_{i=0}^{d} \binom{n}{i}\). Let \(X \subseteq \{0,1\}^n\) with \(|X| < t\). Show that there is a function \(f : \{0,1\}^n \rightarrow \mathbb{R}\) which satisfies:
   - \(f(x) = 0\) for every \(x \in X\).
   - \(f\) is not identically zero.
   - \(\hat{f}(S) = 0\) for every \(S\) with \(|S| > d\).

5. Conclude that for any constant \(k\), any \(k\)-wise independent distribution over \(\{0,1\}^n\) has support size at least \(\Omega(n^{k/2})\).

2 \((k, \varepsilon)\)-bias (15 points)

In class we talked about \(k\)-wise independence and \(\varepsilon\)-bias. It is known that:

- For \(n = 2^t\) and \(k \leq n\) there exists an \(n \times \frac{\log n}{2} k\) matrix over \(\mathbb{F}_2\) such that any \(k\) rows are independent over \(\mathbb{F}_2\).

- There exist an \(\varepsilon\)-biased distributions \(D \subseteq \{0,1\}^m\) with support size \(O(\frac{m}{\varepsilon^2})\).

**Definition 2.** A distribution \(X \subseteq \{0,1\}^n\) is \(k\)-wise \(\varepsilon\)-biased, if for every \(t \leq k\) and \(I = \{i_1, \ldots, i_t\} \subseteq [n]\) the random variable \(X_{i_1} \circ \cdots \circ X_{i_t}\) is \(\varepsilon\)-biased.

Construct a \(k\)-wise \(\varepsilon\)-biased distribution \(X \subseteq \{0,1\}^n\) using \(\log \log n + \log k + 2 \log \frac{1}{\varepsilon} + O(1)\) bits of randomness.
3 Expansion and Codes (25 points)

For a $D$-regular undirected graph $G = (V, E)$ over $N$ vertices:

- $\lambda(G)$ is its second largest eigenvalue in absolute value.
- The vertex expansion of a set $A \subseteq V$ is $\epsilon(A) \overset{\text{def}}{=} \frac{|\Gamma(A)|}{|A|}$ where $\Gamma(A) = \{w \in V : \exists v \in A : (v, w) \in E\}$.

For a subset $A \subseteq V$ the density of $A$ is $\rho(A) \overset{\text{def}}{=} \frac{|A|}{|V|}$.

For a family $G = \{G_n\}_{n \in \mathbb{N}}$ where $G_n$ is a $D$-regular graph over $n$ vertices, we say $G$ is $(\alpha, c)$-expanding if for for all $n$ large enough, and all sets $A$ of density at most $\alpha$, $\epsilon(A) \geq c$.

1. Prove that for every $G = (V, E)$ and $A \subseteq V$,
\[ \epsilon(A) \geq \frac{1}{\rho(A) + (1 - \rho(A))\lambda(G)^2}. \]

2. Let $G = \{G_n\}_{n \in \mathbb{N}}$ be a family of $D$-regular Ramanujan graphs. Show that there exists some constant $\alpha > 0$ such that $G$ is $(\alpha, D/4)$ expanding.

We remark that, in fact, in Ramanujan graphs the true expansion approaches (with $n$) at least $D/2$, and this is tight in the sense that there are Ramanujan graphs with expansion at most $D/2$.

3. Prove that for a fixed $D$, a random $G = \{G_n\}$ of a family of $D$-regular directed graphs, w.h.p., there exists some constant $\alpha > 0$ such that $G$ is $(\alpha, D - 10)$ expanding. (A similar phenomenon is true for undirected graphs, but we need first to define the model of a random $D$-regular undirected graph).

4. Prove that you can never guarantee more than $D - 1$ expansion.

Next, we consider expansion in unbalanced bi-partite graphs with $n$ vertices on the left and $\beta n$ on the right for $0 < \beta < 1$ (thus, in a sense, the graph is condensing). We look at a family $G = \{G_n\}_{n \in \mathbb{N}}$, where $G_n = (V_n, W_n, E_n)$ with left degree $D$, $|V_n| = n$ and $|W_n| = \beta n$. We say $G$ is $(\alpha, c)$-expanding if for all $n$ large enough, and all sets $A \subseteq V$ of density at most $\alpha$, $\epsilon(A) \geq c$.

5. Prove that for a fixed $\beta$ and $D$ a random $G = \{G_n\}_{n \in \mathbb{N}}$ with regular left-degree $D$ is $(\alpha, D - 10)$ expanding, for some constant $0 < \alpha < 1$.

We remark that there are explicit constructions of such graphs with $(\alpha, \frac{2D}{3})$ expansion.

**Definition 3.** Let $G = (V, W, E)$ be a left $D$-regular undirected graph. The code $C(G) \subseteq \{0, 1\}^{|V|}$ is defined as follows:
\[ C(G) = \left\{ x \in \{0, 1\}^{|V|} \mid \forall w \in W : \sum_{v : v \in \Gamma(w)} x_v = 0 \right\} \]

Where addition is done in $\mathbb{F}_2$.

6. Show that $C(G)$ is a linear code with rate at least $(1 - \beta)n$.

7. Prove that if $G$ is $(\alpha, c)$-expanding for $c > D/2$ then $C(G)$ is asymptotically good. Specifically, prove that the distance of $C(G)$ is at least $\alpha n$. 

3
4 Universal traversal sequences (20 points)

**Definition 4.** Let $F$ be a family of $D$-regular labelled graphs. We say the string $\sigma = (\sigma_1, \ldots, \sigma_T) \in [D]^T$ is a universal traversal sequence (UTS) for $F$ if for every graph $G$ in $F$ and every vertex $v$ of $G$, the walk $\sigma$ starting at $v$ will visit all the vertices of the graph.

As usual, let $G = (V, E)$ a $D$-regular, connected, undirected, non-bipartite graph over $n$ vertices, $A_G$ the normalized adjacency matrix of $G$, $\lambda_1 > \lambda_2 > \cdots > \lambda_n$ the spectrum of $A_G$, and $\bar{\lambda} = \max\{-\lambda_n, \lambda_2\}$.

**Fact 5.** $\bar{\lambda} \leq 1 - \frac{1}{2dn^3}$.

1. Let $G$ be the family of all $D$-regular, connected, undirected, non-bipartite graphs over $n$ where $D$ is some constant. Show that there is a polynomial $p(n)$ such that for any $G = (V, E) \in G$ and pair of vertices $s, t \in V$, the probability that a random walk of length $p(n)$ from $s$ does not reach $t$ is at most $2^{-n^{10}}$.

2. Prove that there exists a UTS for $G$ of length $\text{poly}(n)$

5 Eps-bias amplification (25 points)

In this question we construct an $\varepsilon$-biased distribution using some base $\varepsilon_0$-biased distribution and a family of expander graphs.

1. Let $D \subseteq \{0, 1\}^n$ be an $\varepsilon$-biased distribution (i.e. - $D$ is flat over its support and has $\varepsilon$-bias) and let $G = (V, E)$ be a $d$-regular $\lambda$-expander where $|V| = |D|$ and we identify the vertices of $G$ with the elements in $D$.

   We now define a new distribution $D'$ by sampling an ordered edge in $G$ and outputting the sum of its vertices: $D' = \{x_i + x_j \mid (x_i, x_j) \in E\}$.

   Prove that $D'$ is at most $(\lambda + \varepsilon^2)$-biased. What is the support size of $D'$?

2. Suppose $X_0$ is $\varepsilon_0$-biased over $\{0, 1\}^k$ with support size $D_0$, and you have $[N, D, \lambda]$ expanders with $\lambda = \frac{2}{\sqrt{D}}$ for any $N, D$ you wish.

   Prove that by repeating this process $i$ times, you get an $\varepsilon_i = \frac{1}{2}(2\varepsilon_0)^{2i}$-biased distribution over $\{0, 1\}^k$ with support size $\prod_{j=0}^i D_j$ where for any $1 \leq j \leq i$: $D_j = \frac{64}{(2\varepsilon_0)^{2j+2}}$.

3. Use Justesen code and the above recursion to given an $\varepsilon$-biased distribution over $\{0, 1\}^k$ with support size $\tilde{O}(\frac{k}{\varepsilon^8})$, where the $\tilde{0}$ suppresses multiplicative logarithmic terms.

   Good luck!!!