1. Let $A(x,y)$ be a probabilistic algorithm with success probability $\frac{1}{2} + \varepsilon$. $A$ uses $m$ random coins and takes $T$ time. Show how to amplify the success probability to $1 - \delta$ using $m + O(\frac{\log(1/\delta)}{\varepsilon^2})$ random coins and taking $T \cdot \text{poly}(m, \log(1/\delta), \frac{1}{\varepsilon})$ time.

2. (RVW) Let $F$ be a field with $q$ elements. We defined the graph $G = (V = F \times F, E)$ where $((a,b),(c,d)) \in E$ iff $(a,b) \in \ell_{c,d}$ and $\ell_{c,d} = \{(x,y) | y = cx - d\}$. We also defined the map $\text{Rot} : V \times [q] \rightarrow V \times [q]$ by

$$\text{Rot}((a,b),t) = \begin{cases} ((\frac{t}{a}, t - b), t) & \text{If } a \neq 0, t \neq 0 \\ ((t, -b), a) & \text{Otherwise} \end{cases}$$

Prove this defines a rotation map.

Norms and tensors

For a vector $z \in \mathbb{C}^n$ we denote $\|v\|_p = (\sum_{i=1}^n |z_i|^p)^{1/p}$ and $\|z\|_{\infty} = \max_i |z_i|$.

3. (The spectral norm) Let $A$ be a matrix. Define $\|A\| = \sup_{v \neq 0} \frac{\|Av\|_2}{\|v\|_2}$. Prove:

- $\|A + B\| \leq \|A\| + \|B\|$
- $\|cA\| = |c|\|A\|$, $\|A\| = 0$ iff $A = 0$
- $\|AB\| \leq \|A\|\|B\|$
- If $A$ is normal then $\|A\| = \lambda_1(A)$.

4. Let $A$ be a matrix over the complex field $\mathbb{C}$. Define $\|A\|_{\text{row}} = \sup_{v \neq 0} \frac{\|Av\|_{\infty}}{\|v\|_{\infty}}$. Prove:

- $\|A\|_{\text{row}} = \max_i \|A_i\|_1$, where $A_i$ is the $i$'th row of $A$.
- $\|A + B\|_{\text{row}} \leq \|A\|_{\text{row}} + \|B\|_{\text{row}}$
- $\|cA\|_{\text{row}} = |c|\|A\|_{\text{row}}$, $\|A\|_{\text{row}} = 0$ iff $A = 0$
- $\|AB\|_{\text{row}} \leq \|A\|_{\text{row}}\|B\|_{\text{row}}$
- If $A$ is the transition matrix of an undirected graph then $\|A\|_{\text{row}} = 1$.

5. Recall that if $A \in M_{n_1,m_1}(F)$ and $B \in M_{n_2,m_2}(F)$ then $A \bigotimes B \in M_{n_1n_2 \times m_1m_2}(F)$ and is defined by $A \bigotimes B[(i_1,i_2),(j_1,j_2)] = A[i_1,j_1] \cdot B[i_2,j_2]$. Show that:
\[(a)\quad A \otimes B = \begin{pmatrix} a_{1,1}B & a_{1,2}B \\ \vdots & \ddots \\ a_{i,j}B \\ \vdots & \ddots \\ a_{n,n}B \end{pmatrix}\]

(b) Prove that \((A \otimes B)^T = A^T \otimes B^T, (A \otimes B)^\dagger = A^\dagger \otimes B^\dagger\)

c) Prove that the tensor product of two projections is a projection.

d) Prove that the tensor product of two unitary matrices is unitary.

e) Prove that \((A \otimes B) \cdot (C \otimes D) = (AC) \otimes (BD), \) whenever the dimensions fit.

(f) Prove that \(\text{Tr} (A \otimes B) = \text{Tr} (A) \cdot \text{Tr} (B)\).

6. Let \(H = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \). \(\otimes^n H = H \otimes \ldots \otimes H\) denotes the operator obtained by tensoring \(H\) \(n\) times. \(IP\) is the inner product matrix: \(IP[s, t] = (-1)^{(s\cdot t)},\) for any \(s, t \in \{0, 1\}^n\).

(a) Prove \(\otimes^n H = IP\).

(b) Compute the eigenvalues of \(IP\).

(c) Conclude that \(\|IP\| = 2^{n/2}\) and \(\sqrt{\frac{1}{n}} IP\) is unitary.

7. Look at the matrix \(IP\). For two sets \(S, T \subseteq [N = 2^n]\) the combinatorial rectangle \(S \times T\) is the minor of the matrix when restricted to \(S \times T\). The discrepancy \(\Delta(S, T)\) of the combinatorial rectangle is \(|t_1 - t_0|\) where \(t_b\) is the number of times \((-1)^b\) appears in the combinatorial rectangle \(S \times T\). Prove that for any two sets \(S, T, \Delta(S, T) \leq \sqrt{N \cdot |S| \cdot |T|}\).

The eigenvalues and eigenvectors of Cayley graphs

A character of \(G\) is a function \(\chi : G \rightarrow \mathbb{C}^* = \mathbb{C} \setminus 0\) such that for every \(g_1, g_2 \in G\) it holds that \(\chi(g_1 \cdot g_2) = \chi(g_1) \cdot \chi(g_2)\) (where the first multiplication is in \(G\) and the second in \(\mathbb{C}\)). A finite Abelian group has exactly \(|G|\) different characters.

Suppose \(G\) has \(n\) elements, \(G = \{g_1, \ldots, g_n\}\) and \(f : G \rightarrow \mathbb{C}\). We let \(\vec{f}\) denote the vector

\[
\vec{f} = \begin{pmatrix} f(g_1) \\ f(g_2) \\ \vdots \\ f(g_n) \end{pmatrix}
\]

8. For an integer \(k\), let \(\mathbb{Z}_k\) denote the \(\{0, \ldots, k - 1\}\) with addition mod \(k\). Also, let \(w = e^{2\pi i/k}\), i.e., \(w\) is a primitive root of unity of order \(k\). Define \(\chi_1, \ldots, \chi_k\) by \(\chi_i(j) = w^{ij}\).

(a) Prove that \(\chi_1, \ldots, \chi_k\) are characters of \(\mathbb{Z}_k\).

(b) Prove that \(\{\chi_1, \ldots, \chi_k\}\) is an orthogonal basis of \(\mathbb{C}^k\).

9. Let \(G_1, G_2\) be two groups, \(f_1 : G_1 \rightarrow \mathbb{C}\) and \(f_2 : G_2 \rightarrow \mathbb{C}\). Define \(f_1 \otimes f_2 : G_1 \times G_2 \rightarrow \mathbb{C}\) by \(f_1 \otimes f_2(g_1, g_2) = f_1(g_1) \cdot f_2(g_2)\).
• Prove that if $\chi_1$ is a character of $G_1$ and $\chi_2$ a character of $G_2$ then $\chi_1 \otimes \chi_2$ is a character of $G_1 \times G_2$.

• Find all the characters of $\mathbb{Z}_n^2$. Prove their corresponding vectors are orthogonal. How does this compare to Question 6c?

10. Suppose $A \in M_n(\mathbb{C})$ is such that $A_{i,j} = f(g_i g_j^{-1})$ for some function $f : G \to \mathbb{C}$. Prove that $\vec{\chi}$ is an eigenvector of $A$ with eigenvalue $\langle \vec{\chi}, \vec{f} \rangle = \sum_j f(j) \chi(j^{-1})$.

11. Let $G$ be a group and $S \subseteq G$. The Cayley graph $C(G, S)$ is the graph $C(V, E)$ with $V = G$ and $(a, b) \in E$ iff $a = bs^{-1}$ for some $s \in S$. Identify the following graphs:

   • $C(\mathbb{Z}_n, \{1\})$
   • $C(\mathbb{Z}_n, \{1, -1\})$
   • $C(\mathbb{Z}_n^2, \{e_1, \ldots, e_n\})$, where $e_i$ has 1 in the $i$'th coordinate and 0 otherwise.
   • $C(\mathbb{Z}_n \times \mathbb{Z}_n, \{(0, 1), (0, -1), (1, 0), (-1, 0)\})$.

12. Calculate the eigenvalues and the spectral gap of the Cayley graphs given in question 11.