Ex1: DET and low-depth arithmetic and boolean circuits

1. Prove the Schwartz-Zippel lemma.
   If $p : \mathbb{F}^m \to \mathbb{F}$ is a non-zero polynomial of total degree $d$ over a field $\mathbb{F}$ and $\Lambda \subseteq \mathbb{F}$, then $\Pr_{a_1,\ldots,a_m \in \Lambda}[p(a_1,\ldots, a_m) = 0] \leq \frac{d}{|\Lambda|}$.

2. Shortly outline the proof of each of the following:
   (a) Addition of two integers represented in binary is in $\mathsf{AC}^0$?
   (b) Addition of $n$ integers ($n$-bit each) is in $\mathsf{NC}^1$?
   (c) Multiplication of two integers in $\mathsf{NC}^1$?
   (d) Multiplication of two boolean matrices is in $\mathsf{AC}^0$?

3. Prove that $\mathsf{NC}^k \subseteq \mathsf{SPACE}(O(\lg^k n))$. Note the cost of pointers.

4. Prove that $\mathsf{NL} \subseteq \mathsf{AC}^1$. If you use a reduction, carefully note the resources it takes.

5. (Ben-Or) Denote $e_d(x_1, \ldots, x_n) = \sum_{S \subseteq [n], |S| = d} \Pi_{j \in S} x_j, 1 \leq d \leq n$.
   We are going to construct a depth three, polynomial size arithmetic formula for $e_d$ over $\mathbb{C}$ (with addition and multiplication gates of unbounded fan-in). For that:
   - Define $p(t) = p_{x_1,\ldots,x_n}(t) = \Pi_{i=1}^n (t + x_i)$. $p$ is a degree $n$ polynomial $p(t) = \sum_{i=0}^n a_i t^i$. what are the $a_i$ as functions of $x_1, \ldots, x_n$?
   - Build the required circuit.
     Hint: first evaluate $p$ on $n$ points that you choose, then deduce the coefficients of $p$ from the evaluations.
   - Is the family of circuits that you build uniform?

6. Show that matrix inversion of lower triangular matrix is in $\mathsf{SAC}^1$.
   $\mathsf{SAC}^1$: uniform polynomial-size boolean circuits with $O(\log n)$ depth over: unbounded fan-in $\lor$, bounded fan-in $\land$ and $\neg$ at input level only.

7. Show that computing the characteristic polynomial of an arbitrary matrix is in $\mathsf{SAC}^1$.

8. Show that checking the rank of an arbitrary matrix and inverting an invertible matrix are both in $\mathsf{SAC}^1$. 