We know that the minimum distance of a code is related to the packing radius. The Hamming bound was derived by observing the number of non-intersecting balls that could be packed in $\{0,1\}^n$.

A related notion is that of covering radius.

A set $S \subseteq \{0,1\}^n$ has a covering radius $r$ if every point in $\{0,1\}^n$ is contained in some ball $B(z, r)$ for $z \in S$ (i.e., any point in $\{0,1\}^n$ is within distance $r$ from some point in $S$).

Such a set is called a covering code in $\{0,1\}^n$ for radius $r$. For a given set, we can ask what is its minimum covering radius. An interesting, and hard, question is to determine the minimal size of a covering code of radius $r$. Bounds exist relating the covering radius of codes with the minimum distance of its dual.

Intuitively, we shall show that if a code $C$ has distance $d$, then its dual has small 'essential' covering radius.

From this, we shall conclude that the dual code is of large enough size, & hence the code is small.

The essential covering radius is a relaxation where we only demand that most points are covered.

i.e. A set $S \subseteq \{0,1\}^n$ has essential covering radius $r$ if

\[ \left| \bigcup_{z \in S} B(z, r) \right| \geq \frac{1}{n} 2^n \]

i.e. at least a $\frac{1}{n}$ fraction of points are covered by balls of radius $r$ around points in $S$.

Formally, we shall show the following:

* Let $C \subseteq \{0,1\}^n$ be a linear code of min. distance $d$.

Then, for $r = \frac{1}{2} n - \sqrt{d(n-d)} + o(n)$,

\[ \left| \bigcup_{z \in C^\perp} B(z, r) \right| \geq \frac{2^n}{n} \]

i.e. $\frac{1}{2} n - \sqrt{d(n-d)} + o(n)$ is an essential covering radius for the dual of a linear distance-$d$ code.

Once we establish this, then it is easy to derive the MRRW bound for binary linear codes.

* First note that $|C| \leq n |B(z,r)|$ for $r$ as above.

\[ Pf: \quad |C^\perp| |B(z,r)| \geq \left| \bigcup_{z \in C^\perp} B(z, r) \right| \geq \frac{2^n}{n} \]
Also \( |C| \cdot |C^\perp| = 2^n \)

Hence \( |C| \leq n |B(z, r)| \).

\[ (*) \text{ (MRRW bound). } R \leq H_2 \left( \frac{1}{2} - \sqrt{8(1-\delta)} \right) + o(1) \]

\[ \geq n \frac{2^{nH_2 \left( \frac{1}{2} - \sqrt{8(1-\delta)} \right) + o(1)}}{n} \]

(as \( \text{Vol}(n, \text{pt}) \sim 2^n H_2(p) \)).

Taking logarithms & noting that \( \log \frac{n}{n} = o(n) \), we get the result.

Thus, we shall now concentrate on proving that Hamming balls of small radius, around codewords in the dual, essentially cover \( \{01\}^n \).

Note that we can more generally translate any set additively over codewords in the dual, & see when they essentially cover the whole space.

i.e. Suppose \( B \) is some set \( \subseteq \{01\}^n \). We can then consider translates \( z + B \) for \( z \subseteq C^\perp \).

We want some property that \( B \) should satisfy so that

\[ \sum_{z \subseteq C^\perp} (z + B) \geq 2^n \frac{2^n}{n} \]

(Note that \( B \) being a Hamming ball of radius \( r \) is just a special case of this. Once we obtain a general property demanded of \( B \) for essential covering (of the dual), then we can simply show that \( B(z, r) \) satisfies this property for the given \( r \)).

This property is expressed in terms of the maximum eigenvalue of the set \( B \).

For \( B \subseteq \{01\}^n \), define its maximum eigenvalue

\[ \lambda_B = \max \left\{ \frac{\langle Af, f \rangle}{\langle f, f \rangle} \middle| f: \{01\}^n \rightarrow \mathbb{R} \text{, supp}(f) \subseteq B \right\} \]

Here \( A \) is the adjacency matrix of the boolean hypercube graph.

Thus \( \lambda_B \) is the maximal eigenvalue of the adjacency matrix of the subgraph induced by vertices of \( B \).

We shall now show that sets \( B \) with high enough maximal
After that, we shall show that $B(x,n)$ satisfies this max.
eigenvalue condition for the required $n$, & we would be
done.

Let $C \subseteq \{0,1\}^n$ be a linear distance $d$ code, and $B \subseteq \{0,1\}^n$
be such that $\lambda_B \geq n - 2d + 1$.

Then $| \bigcup_{Z \in C^t} (Z + B) | \geq \frac{2^n}{n}$.

for $f_B$ (First note that since $\hat{c}_e(x) = \frac{1}{2^n} c(x)$, if $C$
is of distance $d$, then for $x$ s.t. $wt(x) < d$
$\hat{c}_e(x) = 0$.
(i.e. small weight coefficients in the fourier transform
of the dual are 0).

Also, note the following simple results which shall be
used soon:
- For $g : \{0,1\}^n \rightarrow R$, $(Ag)(x) = \sum_{i=1}^n g(x + e_i)
- A g(x) = 1 \cdot g(x_1) + 1 \cdot g(x_2) + ... + 1 \cdot g(x_n)$
  where $x_1, x_2, ... x_n$ are the $n$ neighbors of $x$.
  Thus $A g(x) = \sum_{i=1}^n g(x + e_i)$.

Suppose $f_B : \{0,1\}^n \rightarrow R$ with $\text{Supp}(f_B) \subseteq B$ is a
maximizer of $\langle Af, f \rangle$ i.e. $\lambda_B = \frac{\langle Af, f \rangle}{\langle f, f \rangle}$
(We shall denote $f_B$ as $f$ and $\lambda_B$ as $\lambda$ from now on)
Then:
- $f$ is non negative $\forall x \in \mathbb{R}$
- $\langle Af, f \rangle$ can only be increased by making negative terms
  positive, for the same $\langle f, f \rangle$, as $A$ is nonnegative
- $\forall x \in \mathbb{R}$, $A f(x) = \lambda f(x)$
  Consider $A^B$ and $f^B$ where we restrict to $B$.
  Then the result follows from Courant-Fischer theorem.
- $\forall x$, $A f(x) \geq \lambda f(x)$. Thus $A f \geq \lambda f$
  For $x \in B$, we showed above.
  For $x \notin B$, $f(x) = 0$, hence the inequality holds.

Now our aim is to show that $| \bigcup_{Z \in C^t} (Z + B) |$ is large
enough.
For this, we shall appropriately define a function $F$ whose
support is contained in \( \bigcup_{z \in \mathbb{C}^d} (z+B) \) and then show that the function's support is large.

To show that a function's support is large, note that a function on small support has a large ratio between its second moment & square of first moment. That is:

By Cauchy-Schwarz Inequality, if \( U \) is its support,

\[
\mathbb{E}[F]^2 = \langle F, 1_U \rangle \leq \mathbb{E}[F^2] \mathbb{E}[1_U] = \frac{|U|}{2^n} \mathbb{E}[F^2]
\]

Hence, we only need to find an \( F \) with \( \text{supp}(F) \) contained in \( \bigcup_{z \in \mathbb{C}^d} (z+B) \) such that

\[
\frac{\mathbb{E}[F^2]}{\mathbb{E}[F]^2} \leq n
\]

We shall define such an \( F \) using \( f \), as follows:

Let \( f_z(x) = f(x+z) \) - a shifted version of \( f \) for each \( z \).

Then define:

\[
F = \frac{1}{2^n} \sum_{z \in \mathbb{C}^d} f_z - 1_{\mathbb{C}^d} * f
\]

Note that \( \text{supp}(f_z) \subseteq z+B \), hence

\[
\text{supp}(F) \subseteq \bigcup_{z \in \mathbb{C}^d} (z+B)
\]

We shall now show that \( \frac{\mathbb{E}[F^2]}{\mathbb{E}[F]^2} \leq n \) by showing that:

\[
\langle AF, F \rangle \geq \lambda \mathbb{E}[F^2]
\]

\[
\langle AF, F \rangle \leq n \mathbb{E}[F^2] + (n-2d) \mathbb{E}[F^2]
\]

Once we show these, then note that:

\[
\lambda \mathbb{E}[F^2] \leq n \mathbb{E}[F^2] + (n-2d) \mathbb{E}[F^2]
\]

\[
\Rightarrow \mathbb{E}[F]^2 \geq \frac{\lambda - (n-2d)}{n} \mathbb{E}[F^2] \geq \frac{\mathbb{E}[F^2]}{n}
\]

Since \( \lambda - (n-2d) \geq 1 \).

Thus, we now only have to show the two bounds on \( \langle AF, F \rangle \).

\[
\mathbb{P}_\delta: \text{For } x \in \{0,1\}^n,
\]

\[
AF(x) = \sum_i F(x + e_i) = \frac{1}{2^n} \sum_i \sum_{z \in \mathbb{C}^d} f(x+z+e_i)
\]

\[
= \frac{1}{2^n} \sum_{z \in \mathbb{C}^d} AF(x+z) \geq \frac{1}{2^n} \sum_{z \in \mathbb{C}^d} \frac{\lambda}{2^n} f(x+z)
\]
Then \( \langle AF \rangle = \mathbb{E}[AF(x)F(x)] \geq \lambda \mathbb{E}[F^2] \)

* \( \langle AF \rangle \leq n \mathbb{E}[F]^2 + (n-2d) \mathbb{E}[F^2] \)

**Proof:** \( \langle AF \rangle = \sum_x \hat{A}F(x) \hat{F}(x) \quad (\text{Parseval Theorem}) \)

Hence we need to study \( \hat{A}F \) and \( \hat{F} \).

First note that:

\[
\hat{F}(\alpha) = \int_{C^d} \hat{f} \quad (\text{Property of Convolutions})
\]

\[
= \frac{|C^d|}{2^n} \int_{C^d} \hat{f}(\alpha)
\]

Hence for \( \alpha \) st. \( \text{wt}(\alpha) < d \), \( \hat{F}(\alpha) = 0 \).

Next consider \( \hat{A}F \).

In fact, for any \( g : \{0,1\}^n \rightarrow \mathbb{R} \),

\[
\hat{A}g(\alpha) = \hat{g}(\alpha) (n-2 \text{wt}(\alpha))
\]

**Proof:** \( \hat{A}g = \sum_i g_{e_i} \)

Hence \( \hat{A}g(\alpha) = \sum_i \hat{g}_{e_i}(\alpha) = \hat{g}(\alpha) \sum_i (-1)^{\langle \alpha, e_i \rangle} \)

(\text{This is because, in general if } g_z(x) = g(x+z) \),

then \( \hat{g}_z(\alpha) = \frac{1}{2^n} \sum_x g_z(x)(-1)^{\langle \alpha, x \rangle} \)

\[
= \frac{1}{2^n} \sum_x g(x+z)(-1)^{\langle \alpha, x \rangle}
\]

\[
= \frac{1}{2^n} \sum_y g(y)(-1)^{\langle \alpha, y \rangle}
\]

\[
= (-1)^{\langle \alpha, z \rangle} \frac{1}{2^n} \sum_y g(y)(-1)^{\langle \alpha, y \rangle}
\]

\[
= (-1)^{\langle \alpha, z \rangle} \hat{g}(\alpha)
\]

Now \( (-1)^{\langle \alpha, e_i \rangle} = (-1)^{\alpha_i} = 1 - 2\alpha_i \)

(\text{where } \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \text{ with } \alpha_i \in \{0,1\}^d \)

Thus \( \hat{A}g(\alpha) = \hat{g}(\alpha) \sum_i (1 - 2\alpha_i) \)

\[
= \hat{g}(\alpha) (n-2 \text{wt}(\alpha))
\]

Thus \( \sum_x \hat{A}F(x) \hat{F}(x) = \sum_x \hat{F}(\alpha)^2 (n-2 \text{wt}(\alpha)) \)

\[
= n \hat{F}(\alpha)^2 + \sum_{\text{wt}(\alpha) \geq d} \hat{F}(\alpha)^2 (n-2 \text{wt}(\alpha))
\]
\[ n \hat{f}(0)^2 + (n-2d) \sum_{x \in B} \hat{f}(x)^2 \leq n \hat{f}(0)^2 + (n-2d) \sum_{x \in B} \hat{f}(x)^2 = n \mathbb{E}[f^2] + (n-2d) \mathbb{E}[f^2] \]

Hence we have proved that for \( B \) such that \( \lambda_B \geq (n-2d+1) \), \( \mathbb{E}[\sum_{x \in B} (z+B)_x] \geq \frac{2n}{n^2} \).

We now need to show that for \( B \) being a Hamming ball of radius \( r \), its maximum eigenvalue is large enough, i.e. \( \lambda_B \geq n-2d+1 \) for appropriate choice of \( r \).

We shall arrive at the right \( r \) so that this is true by showing in general that:

**For** \( B = B(0,r), \lambda_B(n,r) \geq 2 \sqrt{r(n-r)} - o(n) \).

**Pf:** We shall specify a function \( f \) with \( \text{supp}(f) \subseteq B \) and estimate \( \langle A f, f \rangle \) for this \( f \). The \( f \) shall be chosen so that \( \langle f, f \rangle \) this Rayleigh coefficient is huge large enough.

Consider \( f \) as follows:

\[ f : \{0,1\}^n \rightarrow \mathbb{R} \]

\( f \) depends only on the weight of \( x \) and has

\[ \text{support} \{ x : r-M \leq \text{wt}(x) \leq r \} \subseteq B(0,r) \]

where \( M \) is chosen so that \( M = o(n) \) (say \( n^{3/4} \)).

Then \( f(x) = \left\{ \begin{array}{ll} \frac{1}{\sqrt{M}} & \text{if } \text{wt}(x) = i \in [r-M, r] \\ 0 & \text{otherwise} \end{array} \right. \)

We shall now compute \( \langle A f, f \rangle \) and \( \langle f, f \rangle \).

\[ \langle f, f \rangle = \frac{1}{2^n} \sum_{i=r-M}^{r} \sum_{x \in \binom{[n]}{i}} f(x)^2 = \frac{1}{2^n} \sum_{i=r-M}^{r} \binom{n}{i} f(i)^2 = \frac{1}{2^n} (M+1) = \frac{1}{2^n} \left( M + o(1) \right) \]

\[ \langle A f, f \rangle = \frac{1}{2^n} \sum_{x} A f(x) f(x) = \left( \sum_{x} \sum_{y \in \binom{[n]}{i}} A f(x) f(x) \right) \frac{1}{2^n} \]

Now consider an \( x \) of weight \( i \). It has \( i \) neighbors of weight \( i-1 \) and \( n-i \) neighbors of weight \( i+1 \).

Hence \( A f(x) = \sum_{j=i}^{n} f(x + e_j) = i f(i-1) + (n-i) f(i+1) \).
Hence
\[
\sum_{x} Af(x)f(x) = \sum_{i=1}^{n} \binom{n}{i} \left[ f(i) \left( i f(i-1) + (n-i) f(i+1) \right) \right] = \sum_{i=S-M+1}^{n} i \sqrt{\binom{n}{i}} + \sum_{i=S-M}^{n-i} (n-i) \sqrt{\binom{n}{i+1}} = 2 \sum_{i=M+1}^{n} i(n-i+1)
\]

Now note that \(i(n-i+1) \geq (S-M+1)(n-S+M) \geq n(n-1) - o(n^2)\)

Thus \(\langle Af f \rangle \geq \frac{1}{2^n} M(2 \sqrt{n(n-1)} - o(n))\)

\[\Rightarrow \lambda_{B(0,n)} \geq \frac{\langle Af f \rangle}{\langle f f \rangle} \geq 2 \sqrt{n(n-1)} - o(n)\]

We now only need \(n\) such that \(\lambda \geq n-2d+1\)

\[2 \sqrt{n(n-1)} \geq n-2d\]

\[\Rightarrow n = \frac{n}{2} - \sqrt{d(n-d)} + o(n)\]

Thus, putting it all together:

* For \(\gamma = \frac{n}{2} - \sqrt{d(n-d)} + o(n)\), \(B(0,n)\) has maximum eigenvalue

\[\lambda_{B(0,n)} \geq n-2d+1\]

* For any set \(B \subseteq \{0,1\}^n\) s.t. \(\lambda_{B} \geq n-2d+1\), translations of \(B\) over \(C^d\) essentially cover \(\{0,1\}^n\), i.e.

\[\bigcup_{z \in C^d}(z+B) \geq \frac{2^n}{n}\]

Hence

\[\bigcup_{z \in C^d} B(z,\gamma) \geq \frac{2^n}{n}\] for \(\gamma = \frac{n}{2} - \sqrt{d(n-d)} + o(n)\)

* This implies that the rate of the code \(C\) is such that

\[R \leq H_2\left(\frac{1}{2} - \sqrt{b(1-b)}\right) + o(1)\] for \(\delta \leq \frac{1}{2}\)

Thus, we have derived the same MRRW bound for binary linear codes without involving directly Delsarte's LP or orthogonal polynomials.

Also, among all sets \(B \subseteq \{0,1\}^n\), the Hamming ball has maximum second eigenvalue (i.e. we consider a given volume for \(B\))

i.e. \(\lambda_{B} \leq (1 + o(1)) \lambda_{B_{B(0,n)}}\) (Faber-Krahn Maximizer for Hamming cube)