Topic today: Non-uniformity, branching programs, space-bounded computations.

1. (non-uniformity). A language $L$ is in $\text{DTIME}(t(n))|a(n)$ if there exists a machine $M \in \text{DTIME}(t(n))$ and a set $\{a_n \in \{0,1\}^{a(n)}\}$ of advice strings (that depend on the input length but not on the specific input) such that $\forall x, x \in L$ iff $M(x, a_{|x|})$ accepts.

(a) Prove that there exists an undecidable language $L$ in $\text{P}^{1}$.
(b) Prove that $L$ has polynomial size circuits iff $L \in \text{P}^{\text{poly}}$.
(c) Prove that $\text{BPP} \subseteq \text{P}^{\text{poly}}$.
(d) Prove that if $\text{NP} \subseteq \text{P}^{\text{log}}$ then $\text{NP} = \text{P}$
(e) Prove that for every $c > 0$ there exists $L \in \text{PH}$ that requires circuits of size $n^c$.
(f) Prove that if $\text{PSPACE} \subseteq \text{P}^{\text{poly}}$ then $\text{PSPACE} = \text{MA} = \Sigma_2$
    hint: use $\text{IP} = \text{PSPACE}$.

2. ($\text{NC}^k, \text{AC}^k$) We say a language $L$ is in non-uniform $\text{AC}^k$ if there exists a sequence of circuits $\{C_n\}$ solving $L$, with $\{\land, \lor, \neg\}$ gates, unbounded fan-in, $O(\log^k n)$ depth and polynomial size. We say $L$ is in non-uniform $\text{NC}^k$ if in addition the fan-in of the circuits is at most two. The uniform classes are the ones where there exists a uniform machine outputting the $n$’th circuit using only $O(\log(n))$ space.

(a) Prove that $\text{NC}^k \subseteq \text{Space}(O(\log^k n))$ and in particular $\text{NC}^1 \subseteq L$.
(b) Prove that $\text{NL} \subseteq \text{AC}^1$.
(c) Let $f : \{0,1\}^n \rightarrow \{0,1\}^n$. Prove that $f$ has non-uniform polynomial size formulas iff $f$ is in non-uniform-$\text{NC}^1$.
(d) Prove that every problem solvable in non-uniform bounded width polynomial size branching program, is solvable in $\text{NC}^1$.

3. In the following exercises pay attention to the crucial difference between read-once and multiple-access branching programs.

(a) Let $\text{Exactly}_{n/2}$ be the function that gets $n$ boolean values and returns one iff their sum is exactly $n/2$. Find a branching program for the problem with $O(n \log(n))$ length and $O(\log(n))$ width.
(b) Let $\text{Exact} – \text{Clique}_{n/2}$ be the function that gets an undirected graph $G = (V,E)$ on $n$ vertices and returns one iff it is a clique on $n/2$ vertices (and no other edges). Prove that any read-one branching program (of arbitrary width) for the problem requires $2^n/poly(n)$ size.
4. In this exercise we consider multiple-access, space bounded machines. Let $\text{BPL}$ denote the probabilistic, logarithmic-space bounded class with two sided errors (for $x \in L$ the acceptance probability is at least $2/3$, for $x \notin L$ at most $1/3$). Let $\text{RL}$ denote the one-sided error (for $x \in L$ the acceptance probability is at least $1/2$, for $x \notin L$ it is zero) and $\text{ZPL}$ be the zero-sided class (for every input with probability at least half the answer is the correct one, and otherwise it is don’t-know). Let $\text{BPL}^*, \text{RL}^*, \text{ZPL}^*$ be the corresponding classes where the machine has multiple access to the random tape, i.e., the random tape is polynomially long, initialized uniformly at random, read-only, and the tape head can go left or right over it. Similarly $\text{NL}^*$ denotes $\text{NL}$ with multiple access to the witness tape.

(a) Prove that $\text{NL}^* = \text{NP}$.

(b) Prove that $\text{BPL} \subseteq \text{ZPL}^*$
    hint: using Nisan’s generator.

(c) Prove that a pseudo-random generator against small space, multiple access branching program with only a polynomial stretch, suffices to derandomize $\text{BPL}$. 