This worksheet is mandatory. You have to solve at least of the problems and submit it to me by the due date.

Topic today: Graphs and eigenvalues.

Definitions: We assume \( G \) is an undirected \( D \) regular graph with \( n \) vertices, and \( A \) is its adjacency matrix. Thus \( A \) is an \( n \times n \) symmetric matrix. We denote its eigenvector basis by \( v_1, \ldots, v_n \) and the corresponding eigenvalues by \( \lambda_1 \geq \ldots \geq \lambda_n \).

1. (Courant-Fischer equalities) Let \( G \) be as above. Prove that:
   - \( \lambda_1 = \max \{ < Ax, x > : \|x\| = 1 \} \)
   - \( \lambda_2 = \max \{ < Ax, x > : \|x\| = 1, x \perp v_1 \} \).
   - what is \( \lambda_k \)?
   - \( \lambda_n = \min \{ < Ax, x > : \|x\| = 1 \} \).
   - Define \( \bar{\lambda} = \max \{ \lambda_2, -\lambda_n \} \).

2. Let \( G \) be as above. Show that:
   - \( \lambda_1 = d \) and for every \( i, |\lambda_i| \leq d \).
   - \( G \) is connected iff \( \lambda_2 < \lambda_1 \).
   - Suppose \( G \) is connected. \( \lambda_n = -d \) iff \( G \) is bipartite. In that case what is \( v_n \)?

3. Let \( \{ G_n \} \) be a family of degree \( D \) graphs. We saw in class that \( \liminf_{n \to \infty} \lambda(G_n) \geq \sqrt{n} \).
   Prove that \( \liminf_{n \to \infty} \bar{\lambda}(G_n) \geq 2^{1/4} \sqrt{n} \).
   Hint: Look at \( \text{Trace}(A^4) \).

4. (a) (Random walks over expanders converge fast to uniform)
   Let \( G = (V, E) \) be as above. Let \( p_0 \) be some probability distribution over the vertices.
   Let \( p_t \) be the probability distribution after taking \( t \) steps over the graph. Let \( U \) be the uniform distribution over the vertices. Prove that
   \[ |p_t - u|_1 \leq \sqrt{n} \cdot (\frac{\bar{\lambda}}{d})^t \]

   (b) Let \( \text{diam}(G) \) be the diameter of the graph (the largest distance between two vertices).
   Prove that \( \text{diam}(G) \geq \log_{d-1}(n) - 1 \) and \( \text{diam}(G) \leq \frac{3}{2} \log(\frac{d}{\bar{\lambda}}) n + 1 \).

5. Define
   \[ \Phi_G(s) = \min_{S:|S|=s} \frac{|\Gamma(S) \setminus S|}{|S|} . \]

   Let \( G \) be with \( \bar{\lambda}(G) = 2\sqrt{D} \) and \( s \leq \frac{n}{2d} \). Prove that \( \Phi_G(s) \geq \frac{D}{16} \).

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6. (Hoffman) Let $G$ be as above. Let $\alpha(G)$ denote the size of the maximal independent set in $G$. Prove that

- $\alpha(G) \leq \frac{\lambda}{D} |V|$.
- $\alpha(G) \geq \frac{\lambda_{\text{min}}}{\lambda_1-\lambda_{\text{min}}} |V|$.