Homework 1
Due 2006/4/5

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This worksheet is mandatory. You have to solve at least of the problems and submit it to me by the due date.

Topic today: Graphs and eigenvalues.

Definitions: We assume G is an undirected D regular graph with n vertices, and A is its adjacency matrix. Thus A is an $n \times n$ symmetric matrix. We denote its eigenvector basis by $v_1, \ldots, v_n$ and the corresponding eigenvalues by $\lambda_1 \geq \ldots \geq \lambda_n$.

1. (Courant-Fischer equalities) Let G be as above. Prove that:
   - $\lambda_1 = max\{ \ < Ax, x > \ : \ ||x|| = 1 \ \}$
   - $\lambda_2 = max\{ \ < Ax, x > \ : \ ||x|| = 1, \ x \perp v_1 \ \}$.
   - what is λ_k ?
   - $\lambda_n = min\{ \ < Ax, x > \ : \ ||x|| = 1 \ \}$.
   - Define $\overline{\lambda} = MAX\{ \ \lambda_2, -\lambda_n \ \}$. $\bar{\lambda} = max\{ \ | < Ax, x > | \ : \ ||x|| = 1, \ x \perp v_1 \ \}$.
2. Let G be as above. Show that:
   - $\lambda_1 = d$ and for every i, $|\lambda_i| \leq d$.
   - G is connected iff $\lambda_2 < \lambda_1$.
   - Suppose G is connected. $\lambda_n = -d$ iff G is bipartite. In that case what is v_n?
3. Let $\{G_n\}$ be a family of degree D graphs. We saw in class that $\liminf_{n\to\infty} \overline{\lambda}(G_n) \geq \sqrt{n}$. Prove that $\liminf_{n\to\infty} \overline{\lambda}(G_n) \geq 2^{1/4}\sqrt{n}$.

   Hint: Look at $Trace(A^4)$.
4. (a) (Random walks over expanders converge fast to uniform)
   Let $G = (V, E)$ be as above. Let p_0 be some probability distribution over the vertices. Let p_t be the probability distribution after taking t steps over the graph. Let U be the uniform distribution over the vertices. Prove that

$$|p_t - u|_1 \quad \leq \quad \sqrt{n} \ \cdot \ (\frac{\bar{\lambda}}{d})^t$$

   (b) Let $diam(G)$ be the diameter of the graph (the largest distance between two vertices). Prove that $diam(G) \geq \log_{d-1}(n) - 1$ and $diam(G) \leq \frac{3}{2}\log_{(d/\bar{\lambda})} n + 1$.
5. Define

$$\Phi_G(s) \quad = \quad \min_{S:|S|=s} \frac{|\Gamma(S) \setminus S|}{|S|}.$$

   Let G be with $\overline{\lambda}(G) = 2\sqrt{D}$ and $s \leq \frac{n}{2d}$. Prove that $\Phi_G(s) \geq \frac{D}{16}$.

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6. (Hoffman) Let G be as above. Let $\alpha(G)$ denote the size of the maximal independent set in G. Prove that

   - $\alpha(G) \leq \frac{\lambda}{D}|V|$.
   - $\alpha(G) \geq \frac{-\lambda_{min}}{\lambda_1 - \lambda_{min}}|V|$.