Topic today: Extractors, generators and two questions about space bounded computations.

We say a generator \( G : \{0, 1\}^d \rightarrow \{0, 1\}^m \) fools a predicate \( A \), if

\[
\left| \Pr_{x \in \{0, 1\}^m}[A(x) = 1] - \Pr_{y \in \{0, 1\}^d}[A(G(y)) = 1] \right| \leq \varepsilon.
\]

We say \( G \) fools a class of predicates if it fools all members in the class.

1. Prove that for every \( 1 \leq k \leq n \) and every \( 1 \leq m \leq k + d - \log \log(\frac{1}{\varepsilon}) - O(1) \) and \( \varepsilon > 0 \), there exists a \((k, \varepsilon)\) disperser \( F : \{0, 1\}^n \times \{0, 1\}^d \rightarrow \{0, 1\}^m \) with \( d \leq \log(n - k) + \log(\frac{1}{\varepsilon}) + O(1) \).

2. (ISW99, ISW00)

For a predicate \( A \) on \( m \) bits, and \( y \in \{0, 1\}^m \), define a predicate \( A^{\oplus y} \) on \( m \) bits by having \( A^{\oplus y}(x) = A(x \oplus y) \). Define \( A^{\oplus} \) to be the class of all predicates \( A^{\oplus y} \).

Suppose \( P_1, \ldots, P_r \) are distributions on \( \{0, 1\}^m \) and \( A \) a predicate on \( \{0, 1\}^m \). Suppose one of the distributions \( P_i \) \( \varepsilon \)-fools \( A^{\oplus} \). Prove that \( P = P_1 \oplus \ldots \oplus P_r \) \( \varepsilon \)-fools \( A \), where \( P \) is the distribution obtained by sampling independently the distributions \( P_i \) and taking the xor of the samples.

3. (ISW99, ISW00) A \((k, \varepsilon)\) extractor scheme is a function \( G : \{0, 1\}^n \times \{0, 1\}^d \rightarrow \{0, 1\}^m \) such that:

   - for every \( f \in \{0, 1\}^n \) and every predicate \( A \) on \( m \) bits,
   - if \( G^f \) does not \( \varepsilon \)-fool \( A \),
   - then \( K_A(f) \leq k \).

where \( G^f : \{0, 1\}^d \rightarrow \{0, 1\}^m \) is defined by \( G^f(y) = G(f, y) \), and \( K_A(f) \) is the Kolmogorov complexity of the string \( f \) given free access to oracle calls to \( A \).

Prove that

- A \((k, \varepsilon)\) extractor scheme is a \((k + \log(1/\varepsilon), 2\varepsilon)\) extractor.
- Any explicit \((k, \varepsilon)\) extractor is a \((k + O(1), \varepsilon)\) extractor scheme.

4. Savitch sows that \( NL \subseteq DSPACE(O(\log^2 n)) \). Extend the argument to show that \( BPL \subseteq DSPACE(O(\log^2 n)) \).

5. Use Nisan’s generator to show that \( BPL \subseteq DTIMESPACE(poly(n), O(\log^2 n)) \).