Fast Convergence to Nearly Optimal Solutions in Potential Games

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ABSTRACT

We study the speed of convergence of decentralized dynamics to approximately optimal solutions in potential games. We consider α-Nash dynamics in which a player makes a move if the improvement in his payoff is more than an α factor of his own payoff. Despite the known polynomial convergence of α-Nash dynamics to approximate Nash equilibria in symmetric congestion games [7], it has been shown that the convergence time to approximate Nash equilibria in asymmetric congestion games is exponential [25]. In contrast to this negative result, and as the main result of this paper, we show that for asymmetric congestion games with linear and polynomial delay functions, the convergence time of α-Nash dynamics to an approximate optimal solution is polynomial in the number of players, with approximation ratio that is arbitrarily close to the price of anarchy of the game. In particular, we show this polynomial convergence under the minimal liveness assumption that each player gets at least one chance to move in every T steps. We also prove that the same polynomial convergence result does not hold for (exact) best-response dynamics, showing the α-Nash dynamics is required. We extend these results for congestion games to other potential games including weighted congestion games with linear delay functions, cut games (also called party affiliation games) and market sharing games.

Categories and Subject Descriptors

C.2.4 [**Computer-Communication Networks**]: Distributed Systems; F.2.0 [**Analysis of Algorithms and Problem Complexity**]: General; J.4 [**Social and Behavioral Sciences**]: Economics

General Terms

Theory, Economics, Algorithms

Keywords

game theory, Nash equilibrium, price of anarchy, convergence time, potential games, congestion games

1. INTRODUCTION

Computational game theory has lead already to many important insights for understanding Nash equilibria in systems under the control of self-interested agents. Prominent results for the quality of Nash equilibria include bounds on the price of anarchy, which is the ratio between the worst Nash equilibrium and the global optimal solution [23, 10, 26, 21], and for computational complexity [12, 11, 6]. Intuitively, a high price of anarchy a system indicates that it requires a coordination mechanism to achieve good performance. On the other hand, low price of anarchy does not necessarily imply good performance of the system [20, 17]. One main reason for this phenomenon is that in many games with selfish players acting in a decentralized fashion, the repeated selfish behavior of the players may not lead to a Nash equilibrium [17]. Moreover, the convergence rate might be very slow [12]. This motivates the question of whether selfish players acting in a decentralized fashion, converge to approximate solutions in a reasonable amount of time [20, 17, 8, 5].

In this paper, we address this question for the general class of congestion games, which are used to model rout-

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ing, network design and other resource sharing scenarios in distributed systems [23, 2, 16]. We also consider other potential games. In a congestion game there are n players and a set of resources. The strategy of a player consists of a subset of these resources. Each resource possesses a delay function d_e, which depends on the number of players using this resource and the delay(cost) of each player is the sum of the delays associated with his selected resources.

Rosenthal [22] prove that every congestion game has a pure Nash equilibrium, by showing a potential function that is strictly decreasing after any strict improvement of a player. Thus, this property, shows that the natural "Nash Dynamics", in which players iteratively play best response converges to a pure Nash Equilibrium. It has been shown that the problem of finding pure Nash equilibria in congestion games is PLS-complete [12] even with linear latency functions [1]. This result holds even for symmetric congestion games. These results imply examples of congestion games and initial states from which in the Nash dynamics all Nash equilibria have distance exponential in the number of players n. For this reason, Chien and Sinclair [7] study convergence to approximate equilibria in symmetric congestion games. They consider α-Nash equilibria which are states in which no player can decrease his cost by more than a factor of $1-\alpha$ by unilaterally changing his strategy. They investigate the unrestricted α-Nash dynamics, in which we only allow moves that improve the cost of a player by a factor of more than $1-\alpha$ and under the minimal liveness assumption that each player gets at least one chance to move in every T steps. For symmetric congestion games where each resource delay satisfies the "bounded jump assumption", they show that convergence to α-Nash equilibria occurs within a number of steps that is polynomial in the number of players [7]. Recently, Skopalik and Vöcking [25] show examples of asymmetric congestion games with n players and $O(n)$ resources and bounded jump delay functions such that there are states that have distance exponential in the number of players n to all α-Nash equilibria. Thus, the results for convergence to α-Nash equilibria appear in [7] cannot be extended to asymmetric congestion games. These negative results motivate the study of convergence to approximate solutions in asymmetric congestion games.

Mirrokni and Vetta [20] initiated the study of convergence of exact Nash dynamics to approximate solutions and study the unrestricted Nash dynamics in other games. Later, Goemans et al. [17] studied convergence of *random* Nash dynamics for congestion games, and showed polynomial-time convergence to constant-factor solutions in a set of congestion games. We note that their result for random Nash dynamics does not hold for unrestricted Nash dynamics (see Theorem 3.4) and thus, we consider convergence of α-Nash dynamics. Christodoulou et al. [8] study the speed of convergence to approximate solutions in potential games. They show that after a constant number of rounds of α-Nash dynamics the approximation factor of the solution might be a superconstant. They also show that the approximation factor of a state after one round of Nash dynamics is $\Theta(n)$.

Our Results

In this paper, we study the convergence of unrestricted α-Nash dynamics to an approximately optimal solution in different classes of asymmetric congestion games and other potential games. We consider the unrestricted α-Nash dynamics with a *liveness* property that each player gets at least one chance to move in every T steps

We show that for asymmetric congestion games with linear and polynomial delay functions, the unrestricted α-Nash dynamics with a liveness property converges to approximate solutions with approximation ratio of arbitrarily close to the price of anarchy in time that is polynomial in the number of players (for details, see Theorem 3.3 and Remark 3.5). These results are in contrast to the negative results that appear in [12, 1, 25]. We also prove that the same polynomial convergence result does not hold for (exact) best-response dynamics, showing the α-Nash dynamics is required (Theorem 3.4). We extend this result for other potential games. We first extend this result to weighted congestion games with linear delay functions for which we show that any unrestricted α-Nash dynamics satisfying the liveness property converges to a $(2.618+\epsilon)$-approximate solution after polynomial number of α-moves. Furthermore, we extend the results to profit maximizing potential games including cut games (also called party affiliation games) and market sharing games. In these games, players maximize their payoff instead of minimizing their cost. For these games, we need to assume that players play a best-response α-moves, i.e., an α-move that has the maximum possible payoff. For both of these games, we show that any unrestricted α-Nash best-response dynamics satisfying the liveness property converges to a $(2+\epsilon)$-approximate solution after polynomial number of α-moves. This is in contrast to the negative result of Christodoulou et al. [8] for cut games that shows that convergence time of (exact) best-response dynamics to a constant-factor solution in this game is exponential.

Related work

The study of convergence of Nash dynamics is related to local search problems, and PLS-complete problems introduced by Johnson et al. [18]. Fabrikant et al. [12] proved that finding a pure Nash equilibrium of network congestion games is PLS-complete. Ackermann et al. [1] showed that the same problem for network congestion games with linear latency functions is PLS-complete as well. Skopalik and Vöcking [25] showed that finding an approximate Nash equilibrium in congestion games is also PLS-complete.

Mirrokni and Vetta [20] initiated the study of convergence of unrestricted Nash dynamics (also called *covering walks*) to approximate solutions in the context of valid-utility games [26] and did not consider α-Nash dynamics. Motivated by studying the Nash dynamics and convergence to approximate solutions, Goemans et al. [17] introduced sink equilibria, and proved that in weighted congestion games, random Nash dynamics converges to a constant-factor approximately optimal solution in expected polynomial time. However, they do not provide any bound for the convergence time of deterministic unrestricted Nash dynamics. In fact, in Theorem 3.4, we show a lower bound for deterministic Nash dynamics for these games, showing that the above result only holds for random Nash dynamics. Christodoulou et al. [8] showed a tight bound of $\Theta(n)$ for the approximation factor of the solution after one round of α-Nash dynamics in congestion games with linear latency functions. They also showed that for congestion games with linear latency functions, after a constant rounds of Nash dynamics, players may not converge to an approximate solution (for more results, see [13]). Here, we show that after a polynomial rounds of α-Nash dynamics, players converge to a constant-factor solution. Chekuri et al. [5] and Charikar et al. [4] studied convergence of Nash dynamics to approximate solutions in network cost sharing games (for more results, see, e.g, [14]).

The study of α-moves for convergence to approximate solutions has been also considered by Christoulou [8] in the context of cut games. They show that for any constant α (and not for an $\alpha = o(1)$) after one round of α-moves of players in a cut game, the value of the cut is a constant-factor approximate solution. Their proof does not handle the convergence of unrestricted dynamics. For a more complete list of results in these areas, see Mirrokni [19].

Cut Games (or party affiliation games) are potential games defined on an edge-weighted graph [12, 24, 8]. Nash dynamics for these games correspond to the local search algorithm for the Max-Cut problem. Schaffer and Yannakakis [24] proved that finding a Nash equilibrium in this game is PLS-complete. Christodoulou et al. [8] showed an exponential lower bound for the convergence time of (exact) best-response dynamics to constant-factor approximate solutions in these games. In contrast, we show polynomial convergence of α-Nash best-response dynamics in these games. Market sharing games are a special case of profit maximizing congestion games and valid-utility games [26] that has been studied for the content distribution in service provider networks [16]. Mirrokni and Vetta [20] show that after one round of best responses in which each player get exactly one chance to play best response, players reach an $O(\log n)$-approximate solution.

2. PRELIMINARIES

2.1 General Definitions

Strategic games. A strategic game (or a normal-form game) $\Lambda =< N, (\Sigma_i), (u_i) >$ has a finite set $N = \{1, \ldots, n\}$ of players. Player $i \in N$ has a set Σ_i of actions (or strategies). We call a game *symmetric* if all players share the same set of strategies, otherwise we call it *asymmetric*. The joint action set is $\Sigma = \Sigma_1 \times \cdots \times \Sigma_n$ and a joint action $S \in \Sigma$ is also called a *profile* or *strategy profile*. The payoff function of player i is $u_i : \Sigma \to \mathbb{R}$, which maps the joint action $S \in \Sigma$ to a real number. Let $S = (S_1, \ldots, S_n)$ denote the profile of actions taken by the players, and let $S_{-i} = (S_1, \ldots, S_{i-1}, S_{i+1}, \ldots, S_n)$ denote the profile of actions taken by all players other than player i. Note that $S = (S_i, S_{-i})$. An *improvement* move S'_i for a player i in a profile S is a move for which $u_i(S_{-i}, S'_i) \geq u_i(S)$. A *best response* move S''_i for a player i in a profile S is an improvement move that has the maximum payoff. In this paper, we consider two types of games: *cost minimizing games* and *profit maximizing games*. In cost minimizing games, each player i wants to minimize the cost $c_i(S) = -u_i(S)$ in strategy profile S. This type of games include congestion games with polynomial latency functions. In profit maximizing games, each player i wants to maximize the profit $p_i(S) = u_i(S)$ in strategy profile S. This type of games include market sharing games and cut games.

Nash equilibria (NE): A joint action $S \in \Sigma$ is a *pure Nash equilibrium* if no player $i \in N$ can benefit from unilaterally deviating from his action to another action, i.e., $\forall i \in N \; \forall S'_i \in \Sigma_i \; : \; u_i(S_{-i}, S'_i) \leq u_i(S)$. We can also define α-Nash equilibria as follows. For $1 > \alpha > 0$, a state S is an α-Nash equilibrium if for every player i, $c_i(S_{-i}, S'_i) \geq (1-\alpha)c_i(S)$ for all $S'_i \in \Sigma_i$.

State graph. Given any game Λ, the state graph $G(\Lambda)$ is an arc-labelled directed graph as follows. Each vertex in the graph represents a joint action S. There is an arc from state S to state S' with label i iff there exists player i and action $S'_i \in \Sigma_i$ such that $S' = (S_{-i}, S'_i)$, i.e., S' is obtained from S by a move of a single player i that improves his payoff from S to S'.

Exact potential games. A game is called an *exact potential game* if there is a function ϕ such that for any edge of the state graph (S, S') with deviation of player i, we have $\phi(S') - \phi(S) = u_i(S') - u_i(S)$. We denote the minimal potential of the game by ϕ^*.

Social function. Given any game Λ, in order to measure the performance of strategy profiles of players, we define a social function for any strategy profile S. This social function for minimizing cost games is denoted by $cost(S)$ and we denote by $OPT(\Lambda)$ the minimal social cost of a game Λ. i.e., $OPT(\Lambda) = \min_{S\in\Sigma} cost_\Lambda(S)$. We denote by $cost_Z(S)$, the sum of the payoffs of the players in the set Z, when the game Λ is clear from the context, i.e., $cost_Z(S) = \sum_{i\in Z} c_i(S)$. For profit maximizing games, the social function is denoted by $profit(S)$ and we denote by $OPT(\Lambda)$ the maximal social cost of a game Λ. i.e., $OPT(\Lambda) = \max_{S\in\Sigma} profit_\Lambda(S)$. We denote by $profit_Z(S)$, the sum of the payoffs of the players in the set Z, when the game Λ is clear from the context, i.e., $profit_Z(S) = \sum_{i\in Z} p_i(S)$.

α-Nash dynamics. For $0 < \alpha \leq 1$, this dynamics allows only *α-moves* of the players, where α-move of a player is a move that improves his cost by a factor more than $1 - \alpha$, i.e., if player i moves from action S_i to action S'_i then $c_i(S_{-i}, S'_i) < (1-\alpha)c_i(S)$. We consider the *unrestricted* α-Nash dynamics with *liveness* property, which allows an adversary to order the players moves in each round as long as every player has at least one chance to move in each round. The liveness property requirement is that in each interval of length T every player appears at least once. For profit maximizing games, an α-move is a move that increases the payoff by a factor more than $1 + \alpha$. In these games, we study α-Nash dynamics under the assumption that players play a best response when they get a chance. We call this dynamics, the α-Nash best-response dynamics. Also, an *α-Nash best-response move* is a best response α-move.

α-Nash best-response dynamics is also considered by [19, 8] (called $1+\alpha$-greedy players). The liveness property have been considered by [7] and [20]. Mirrokni and Vetta [20] call a round in which each player gets at least a chance to move, a *covering walk*.

Nice Potential Games. Consider a potential game Λ. Let S be a profile of the players and let S'_i be the best response for any player i. For each player i, let $\Delta_i(S) = c_i(S) - c_i(S_{-i}, S'_i)$ and let $\Delta(S) = \sum_i \Delta_i(S)$. Also, for any set of players Z, let $\Delta_Z(S) = \sum_{i\in Z} \Delta_i(S)$. We may drop the (S) part of the terms and denote these terms by Δ_i and Δ_Z, if the profile is determined clearly in the context. The exact potential games considered in this work has the following nice property, which relates the social cost of a state S to the optimal cost by the total gain of the individual best responses of the players, where β is the price of anarchy of the game.

Definition 2.1. *An exact potential game Λ with potential function ϕ and social function cost is β-*nice *iff for any* *state S, it holds that (i) $cost(S) \leq \beta OPT(\Lambda) + 2\Delta(S)$, and (ii) $\phi(S) \leq cost(S)$.*

We show that the α-Nash dynamics converges in polynomial time to a state S with $\Delta(S)$ that is arbitrarily close to zero. Therefore the approximation ratio of the solution S is arbitrarily close to the price of anarchy.

Bounded Jump Property.

DEFINITION 2.2. *(γ-Bounded Jump). For any value $\gamma \geq 1$, a game Λ satisfies the γ-bounded jump condition if for every profile S and every player i with improvement move S'_i, it holds that*

1. *for every player j, $c_j(S) - c_j(S_{-i}, S'_i) \leq c_i(S)$.*
2. *for every improvement action S'_j of player j, it holds $c_j(S_{-\{i,j\}}, S'_i, S'_j) - c_j(S_{-j}, S'_j) \leq \gamma \cdot c_i(S_{-i}, S'_i)$.*

Lemma 4.10 shows that congestion games with resources that satisfy the γ-bounded jump condition, studied in [7, 25], satisfy the γ-bounded jump property according to definition 2.2. Therefore it is sufficient to assume the bounded jump property according to definition 2.2 for this class of games.

ε-approximate α-equilibria. Given a strategy profile S, we call the set of players that cannot make an α-move, α-*equilibrium* players.

DEFINITION 2.3. *A state S is an ε-approximate α-equilibrium if $\Delta_O(S) \leq \varepsilon \cdot cost(S)$ where O is the set of players that can play an α-move.*

2.2 Cost Minimizing Congestion Games

In this part, we define cost minimizing congestion games. Since the focus of this paper is on these games, and for brevity, we call these games, congestion games.

Unweighted Congestion Games. An unweighted congestion game is defined by a tuple $< N, E, (\Sigma_i)_{i\in N}, (d_e)_{e\in E} >$ where E is a set of facilities, $\Sigma_i \subseteq 2^E$ the strategy space of player i, and $d_e : \mathbb{N} \rightarrow \mathbb{Z}$ a delay function associated with resource e. For a joint action S, we define the *congestion* $n_e(S)$ on resource e by $n_e(S) = |\{i|e \in S_i\}|$, that is $n_e(S)$ is the number of players that selected an action containing resource e in S. The cost $c_i(S)$ of player i in a joint action S is $c_i(S) = -u_i(S) = \sum_{e\in S_i} d_e(n_e(S))$. [22] showed that every congestion game possesses at least one pure Nash equilibrium by considering the potential function $\phi(S) = \sum_e \sum_{i=1}^{n_e(S)} d_e(i)$.

Weighted Congestion Games. In *weighted congestion games*, player i has weighted demand w_i. We denote by $l_e(S)$, the congestion(load) on resource e in a state S, i.e., $l_e(S) = \sum_{i|e\in S_i} w_i$. The cost of a player in a state S is $c'_i(S) = \sum_{e\in S_i} d_e(l_e(S))$. The total cost is the weighted sum $cost(S) = \sum_{i\in N} w_i c'_i(S) = \sum_{e\in E} l_e d_e(l_e(S))$. Note that congestion games is a special case of weighted congestion games with $w_i = 1$ for every player i. [15] showed that every weighted congestion game with linear latency functions possesses at least one pure Nash equilibrium by considering a potential function equivalent to

$\phi(S) = \frac{1}{2}\left(\sum_e l_e(S) d_e(l_e(S)) + \sum_i \sum_{e\in S_i} w_i d_e(w_i)\right)$. We use the fact that this potential function is an exact potential function if the cost of a player in a state S is $w_i c'_i(S)$. To simplify the presentation of the results we assume that the cost of any player i in a state S is $c_i(S) = w_i c'_i(S)$.

2.3 Profit Maximizing Congestion Games

Cut Games. Cut game is a profit maximizing congestion game that is defined on an edge-weighted undirected graph $G(V,E)$, with n vertices and edge weights $w : E(G) \rightarrow \mathbb{Q}^+$. We assume that G is connected, simple, and does not contain loops. For each $v \in V(G)$, let $\deg(v)$ be the degree of v, and let $\mathsf{Adj}(v)$ be the set of neighbors of v. Let also $w_v = \sum_{u\in \mathsf{Adj}(v)} w_{uv}$. A cut in G is a partition of $V(G)$ into two sets, T and $\bar{T} = V(G) - T$, and is denoted by $(T, \bar{T})$. The value of a cut is the sum of edges between the two sets T and $\bar{T}$, i.e $profit(S) = \sum_{v\in T, u\in \bar{T}} w_{uv}$.

The *cut game* on a graph $G(V,E)$, is defined as follows: each vertex $v \in V(G)$ is a player, and the strategy of v is to choose one side of the cut, i.e. v can choose $S_v = -1$ or $S_v = 1$. A strategy profile $S = (S_1, S_2, \ldots, S_n)$, corresponds to a cut $(T, \bar{T})$, where $T = \{i|S_i = 1\}$. The payoff of player v in a strategy profile S, denoted by $p_v(S)$, is equal to the contribution of v in the cut, i.e. $p_v(S) = \sum_{i:S_i \neq S_v} w_{iv}$. It follows that the cut value is equal to $\frac{1}{2}\sum_{v\in V} p_v(S)$. If S is clear from the context, we use p_v instead of $p_v(S)$ to denote the payoff of v. We denote the maximum value of a cut in G, by $c(G)$. These games are exact potential games, and the potential function is $\phi(S) = profit(S) = \sum_{v\in T, u\in \bar{T}} w_{uv}$.

Market Sharing Games. A market sharing game is defined by a tuple $< N, M, (\Sigma_i)_{i\in N}, (v_j)_{j\in M} >$ where M is a set of markets, $\Sigma_i \subseteq 2^M$ the strategy space of player i, and v_j the value of market j. For a joint action S, we define the *congestion* $n_j(S)$ on market j by $n_j(S) = |\{i|j \in S_i\}|$, that is $n_j(S)$ is the number of players that selected an action containing market j in S. The payoff $p_i(S)$ of player i in a joint action S is $p_i(S) = u_i(S) = \sum_{j\in S_i} \frac{v_j}{n_j(S)}$. Market sharing games are maximization congestion games with potential function $\phi(S) = \frac{1}{\log n}\sum_{j\in M}\sum_{i=1}^{n_j(S)} \frac{v_j}{i}$. The social function is the sum of payoff of players or the total value of the market satisfied, i.e., $profit(S) = \sum_{i\in N} p_i(S) = \sum_{j\in \cup_{i\in N} S_i} v_j$.

3. CONVERGENCE OF THE α-NASH DYNAMICS

In this section, we consider the unrestricted α-Nash dynamics with a liveness property for nice exact potential games satisfying the bounded jump property. Throughout this section, let C be the set of α-equilibrium players and let O be the set of all other players, i.e., the players that can make an α-move. First we observe the following simple lemma.

LEMMA 3.1. *If a state S is in an ε-approximate α-equilibrium, then $\Delta(S) \leq (\alpha + \varepsilon)cost(S)$.*

PROOF. Since C is the set of α-equilibrium players, $\Delta_C(S) \leq \alpha \cdot cost_C(S)$. Thus, $\Delta(S) = \Delta_C(S) + \Delta_O(S) \leq (\alpha + \varepsilon)cost(S)$. □

As a warmup example, we consider a (restricted) *basic dynamics*, where in each step, among all players that can play an α-move, we choose the player with the maximum absolute improvement, and let him move.

LEMMA 3.2. *Let $\frac{1}{8} > \delta \geq \alpha$. Consider an exact potential game Λ that satisfies the β-nice property and any initial state S_{init}. The basic dynamics generates a profile S with*

$cost(S) \leq \beta(1+O(\delta))OPT(\Lambda)$ in at most $O\left(\frac{n}{\delta}\log(\frac{\phi(S_{init})}{\phi^*})\right)$ steps.

Proof. Consider a step that starts with profile S. Let $\varepsilon_O = \Delta_O(S)/cost(S)$. By definition 2.3 the state S is an ε_O-approximate α-equilibrium. Now, there are two cases:
Case 1: $\varepsilon_O \leq \delta$. It follows from Lemma 3.1 that $\Delta(S) \leq (\alpha + \varepsilon_O)cost(S) \leq (\alpha+\delta)cost(S)$. Hence, by definition 2.1, the dynamics reached $\beta(1+4(\alpha+\delta))$-approximation of the optimal cost.
Case 2: $\varepsilon_O > \delta$. It follows that $\Delta_O(S) > \delta \cdot cost(S)$. Hence, there exists a player $j \in O$ such that $\Delta_j(S) > \frac{\delta}{n}cost(S)$. Thus, $\Delta_j(S) > \frac{\delta}{n}\phi(S)$, since $\phi(S) \leq cost(S)$. Therefore the potential gain is at least $\frac{\delta}{n}\phi(S)$. Let $\phi(t)$ denote the potential in step t. Then, $\phi(t) \leq \phi(S_{init})(1-\frac{\delta}{n})^t$. Since $\phi(t) \geq \phi^*$, the upper bound on the number of steps follows. □

The above basic Nash dynamics requires some coordination that chooses the player with the maximum gain at each step. Now we show similar results for unrestricted Nash dynamics.

Theorem 3.3. *Let $\frac{1}{8} > \delta \geq 4\alpha$. Consider an exact potential game Λ that satisfies the β-nice property and the γ-bounded jump condition. For any initial state S_{init}, the unrestricted α-Nash dynamics with liveness property generates a profile S with $cost(S) \leq \beta(1+O(\delta))OPT(\Lambda)$ in at most $O\left(\frac{\gamma n}{\alpha\delta}\log(\frac{\phi(S_{init})}{\phi^*})\cdot T\right)$ steps.*

Before proving Theorem 3.3 we point out that the α-Nash dynamics is necessary for polynomial time convergence to nearly optimal solutions for nice exact potential games satisfying the bounded jump property, that is, we show that even after exponentially many steps, the unrestricted exact Nash dynamics with a liveness property for asymmetric congestion games with linear delay functions, which are 2.5-nice and 2-bounded jump as we show in section 4, may generate strategy profiles whose social cost is far from the optimal solution.

Theorem 3.4. *There exists an exact potential game Λ that satisfies the β-nice property and the γ-bounded jump condition ($\beta = 2.5$ and $\gamma = 2$), and an initial state S_{init} from which the unrestricted exact best-response dynamics with liveness property generates a profile S with $cost(S) \geq \Omega(\frac{\sqrt{n}}{\log n})OPT$ after an exponential number of steps. In particular, this holds for a congestion game with linear latency functions.*

The proof of this theorem is based on constructing a long involved example with several components, and is left to the full version. We now present the proof of Theorem 3.3.

Proof. Let $\alpha' = 4\alpha$. It is sufficient to consider the case that the players are not in a δ-approximate α'-equilibrium, since otherwise it follows from Lemma 3.1 and Definition 2.1 that the dynamics reached a $\beta(1+4(\alpha'+\delta))$-approximation of the optimal cost. We show that in each interval of T steps the potential decreases by a factor of at least $\frac{\alpha\delta}{4\gamma n}$. Let $S^0, S^1, \ldots, S^T$ denote the joint actions of the players in times $0, 1, \ldots, T$ of this interval respectively. Since S^0 is not a δ-approximate α'-equilibrium, there exists a player with an improvement α'-move. Consider player j with the maximum absolute improvement α'-move and let S'_j be his best response. Recall that $\Delta_j(S^0) = c_j(S^0) - c_j(S^0_{-j}, S'_j)$. Let $\Delta'_j = \Delta_j(S^0)$ and let t' be the first time in this interval that player j is allowed to move. We denote by U the set of times before time t', where players made α-moves and we denote by $w(t)$ the player that moved at time t for each $t \in U$. Let $A = \sum_{t\in U} c_{w(t)}(S^t)$ be the sum of the costs of the moving players when they make their moves. Now, we consider two cases:
Case 1: $A \leq \frac{\Delta'_j}{4\gamma}$. By the first condition of the bounded jump property, we have for each $t \in U$

$$c_j(S^t) - c_j(S^{t+1}) \leq c_{w(t)}(S^t). \quad (1)$$

Summing over all times $t \in U$, we obtain:

$$c_j(S^0) - c_j(S^{t'}) \leq \sum_{t\in U} c_{w(t)}(S^t) = A \leq \frac{\Delta'_j}{4\gamma} \leq \frac{\Delta'_j}{4}. \quad (2)$$

Where the first inequality follows since the sum of the left hand side of equation (1) telescopes.

Similarly, by the second property of the bounded jump assumption, we obtain

$$c_j(S^{t'}_{-j}, S'_j) - c_j(S^0_{-j}, S'_j) \leq \gamma \cdot A \leq \gamma\frac{\Delta'_j}{4\gamma} \leq \frac{\Delta'_j}{4}. \quad (3)$$

By summing inequalities (2) and (3), we get

$$\begin{aligned} c_j(S^{t'}) - c_j(S^{t'}_{-j}, S'_j) &\geq c_j(S^0) - c_j(S^0_{-j}, S'_j) - \frac{\Delta'_j}{2} \\ &= \Delta'_j - \frac{\Delta'_j}{2} = \frac{\Delta'_j}{2}. \end{aligned}$$

By the second property of the bounded jump assumption we also get

$$c_j(S^{t'}) \leq c_j(S^0) + \gamma \cdot A \leq c_j(S^0) + \gamma\frac{\Delta'_j}{4\gamma} = c_j(S^0) + \frac{\Delta'_j}{4}. \quad (4)$$

Hence,

$$c_j(S^{t'}) \leq c_j(S^0) + \frac{\Delta'_j}{4} < \frac{\Delta'_j}{\alpha'} + \frac{\Delta'_j}{4} < 2\frac{\Delta'_j}{4\alpha} = \frac{\Delta'_j}{2\alpha}.$$

Where the second inequality follows from the fact that Δ'_j is the improvement of player j when making his best response, which is an α'-move in step 0. Thus, $\alpha \cdot c_j(S^{t'}) < \frac{\Delta'_j}{2}$. As a result, using this inequality and inequality (4), we get $\alpha \cdot c_j(S^{t'}) < c_j(S^{t'}) - c_j(S^{t'}_{-j}, S'_j)$. Therefore, player j can make an α-move at time t' and decrease the potential ϕ by at least $\alpha \cdot c_j(S^{t'}) \geq \alpha\frac{\Delta'_j}{2} \geq \frac{\alpha\delta}{2n}\phi(S^0)$.
Case 2: $A > \frac{\Delta'_j}{4\gamma}$. Since A is the sum of the costs of players making an α-move when making the move, these players decrease the potential ϕ by at least $\alpha A > \frac{\alpha\Delta'_j}{4\gamma} \geq \frac{\alpha\delta}{4\gamma n}\phi(S^0)$.

Let $\phi(i)$ denote the potential in round i. Then, in both cases $\phi(i) \leq \phi(S_{init})(1-\frac{\alpha\delta}{4\gamma n})^i$. Since $\phi(i) \geq \phi^*$, the upper bound on the number of steps follows. □

Remark 3.5. The above theorem shows that we reach a state with cost at most $\beta(1+O(\delta))$ of the optimum after polynomial number of α-moves. Eventhough after this state the cost of solutions can increase, it follows from the proof of the theorem that the number of states in which the cost of the solution is more than a $\beta(1+O(\delta))$-approximation is at most $O(\frac{\gamma n}{\alpha\delta}\log(\frac{\phi(S_{init})}{\phi^*})T)$. In addition, since the potential function is always decreasing after any α-move, the cost can increase by a factor of at most $\frac{cost(S)}{\phi(S)}$. It is not hard to show that the ratio $\frac{cost(S)}{\phi(S)}$ for any strategy profile in congestion games with polynomial delay functions of degree d is at most $O(d)$ and for weighted congestion games with linear functions is at most $O(1)$. As a result, for both type of congestion games that we consider in Section 4, the cost of any state after a polynomial number of steps reach a constant-factor approximate solution and remains within a constant factor of the optimal solution.

4. CONGESTION GAMES

In this section we consider weighted congestion games with linear latency functions and congestion games with linear and polynomial latency functions.

4.1 Linear Latency Functions

In this section we consider weighed and unweighted congestion games with linear latency functions. Specifically $d_e(x) = a_e x + b_e$ for each resources $e \in E$, where a_e and b_e are nonnegative reals. For simplicity we only consider the identity function $d_e(x) = x$. It is easy to verify that all the proofs work for the general case as well.

4.1.1 Weighted Congestion Games

We first show that weighted congestion games with linear latency functions are *β-nice* according to definition 2.1 with $\beta = \frac{3+\sqrt{5}}{2} \approx 2.618$.

LEMMA 4.1. *Congestion games with linear latency functions are β-nice potential games with $\beta = \frac{3+\sqrt{5}}{2}$.*

The proof require the following two Lemmas. The first lemma appears in [3] and the second lemma is a simple fact.

LEMMA 4.2. *Consider a weighted congestion game Λ with linear delay functions. Let S be any profile and S^* be a profile of the optimal solution, then*

$$\sum_i c_i(S_{-i}, S_i^*) \leq \sqrt{cost(S)}\sqrt{cost(S^*)} + cost(S^*).$$

LEMMA 4.3. *For every pair of nonnegative integers x, y, if $x^2 \leq x + 1 + y$, then $x^2 \leq \frac{3+\sqrt{5}}{2} + 2y$.*

PROOF. Let S^* be a profile of the optimal solution and let S be any profile. Applying Lemma 4.2, we get $\sum_i c_i(S_{-i}, S_i^*) \leq \sqrt{cost(S)}\sqrt{cost(S^*)} + cost(S^*)$. Note that $cost(S) - \sum_i c_i(S_{-i}, S_i^*) \leq \Delta(S)$, since for any player i with best response S_i', $c_i(S_{-i}, S_i') \leq c_i(S_{-i}, S_i^*)$. Thus, by adding $\Delta(S)$ to both sides of the inequality, we get $cost(S) \leq \sqrt{cost(S)}\sqrt{cost(S^*)} + cost(S^*) + \Delta(S)$. Let $x = \sqrt{\frac{cost(S)}{cost(S^*)}}$ and let $y = \frac{\Delta(S)}{cost(S^*)}$. Now, we divide the above inequality by $cost(S^*)$ and express the result in terms of x and y. Thus, $x^2 \leq x+1+y$. Applying Lemma 4.3, we get $x^2 \leq \frac{3+\sqrt{5}}{2} + 2y$. This completes the proof of the Lemma.

□

Next we show that weighted congestion game with linear delay functions satisfy the 1-bounded jump condition.

LEMMA 4.4. *Let Λ be a weighted congestion game with linear delay functions. Then, the game Λ satisfies the 1-bounded jump condition according to definition 2.2.*

PROOF. Consider any profile S and any player i with improving action S_i'. We first show the first property in definition 2.2. Consider any player j. Then,

$$\begin{aligned} & c_j(S) - c_j(S_{-i}, S_i') \\ \leq\ & w_j \sum_{e \in (S_i \setminus S_i') \cap S_j} l_e(S) - (l_e(S) - w_i) \\ =\ & w_j \sum_{e \in (S_i \setminus S_i') \cap S_j} w_i = w_i \sum_{e \in (S_i \setminus S_i') \cap S_j} w_j \\ \leq\ & w_i \sum_{e \in (S_i \setminus S_i') \cap S_j} l_e(S) \leq w_i \sum_{e \in S_i} l_e(S) \\ =\ & c_i(S). \end{aligned}$$

For the second property in definition 2.2. Consider any player j with action S_j'. Then,

$$\begin{aligned} & c_j(S_{-\{i,j\}}, S_i', S_j') - c_j(S_{-j}, S_j') \\ \leq\ & w_j \sum_{e \in (S_i' \setminus S_i) \cap S_j'} (l_e(S_{-j}, S_j') + w_i) - l_e(S_{-j}, S_j') \\ =\ & w_j \sum_{e \in (S_i' \setminus S_i) \cap S_j'} w_i = w_i \sum_{e \in (S_i' \setminus S_i) \cap S_j'} w_j \\ \leq\ & w_i \sum_{e \in (S_i' \setminus S_i) \cap S_j'} l_e(S_{-i}, S_i') \leq w_i \sum_{e \in S_i'} l_e(S_{-i}, S_i') \\ =\ & c_i(S_{-i}, S_i'). \end{aligned}$$

□

Theorem 3.3 and Lemmas 4.1, 4.4 yield the following corollary.

COROLLARY 4.5. *Let $\frac{1}{8} > \delta \geq \alpha$. Consider a weighted congestion game Λ with linear latency functions and any initial state S_{init}. The unrestricted α-Nash dynamics with liveness property generates a profile S with $cost(S) \leq \frac{3+\sqrt{5}}{2}(1 + O(\delta))OPT(\Lambda)$ in at most $O\left(\frac{n}{\alpha\delta}\log(\frac{\phi(S_{init})}{\phi^*}) \cdot T\right)$ steps.*

4.1.2 Unweighted Congestion Games

We first show that congestion games with linear latency functions are *β-nice* according to definition 2.1 with $\beta = 2.5$.

LEMMA 4.6. *Congestion games with linear latency functions are β-nice potential games with $\beta = 2.5$.*

The proof requires the following two Lemmas which appear in [9].

LEMMA 4.7. *Consider a congestion game Λ with nonnegative, non-decreasing delay functions. Let S be any profile and let S^* be a profile of the optimal solution, then*

$$\sum_i c_i(S_{-i}, S_i^*) \leq \sum_{e \in E} n_e(S^*) d_e(n_e(S) + 1).$$

LEMMA 4.8. *For every pair of nonnegative integers x, y, it holds $x(y+1) \leq \frac{5}{3}x^2 + \frac{1}{3}y^2$.*

PROOF. Let S^* be a profile of the optimal solution and let S be any profile. Applying Lemma 4.7, we obtain $\sum_i c_i(S_{-i}, S_i^*) \leq \sum_{e \in E} n_e(S^*) d_e(n_e(S) + 1)$. Applying

Lemma 4.8, we get

$$\begin{aligned}\sum_i c_i(S_{-i}, S_i^*) &\leq \sum_{e\in E}\left(\frac{5}{3}n_e(S^*)^2 + \frac{1}{3}n_e(S)^2\right)\\ &= \frac{5}{3}\sum_{e\in E} n_e(S^*)^2 + \frac{1}{3}\sum_{e\in E} n_e(S)^2\\ &= \frac{5}{3}cost(S^*) + \frac{1}{3}cost(S).\end{aligned}$$

Recall that $cost(S) - \sum_i c_i(S_{-i}, S_i^*) \leq \Delta(S)$, where S_i' is the best response of any player i. Thus, by multiplying the inequality by 3/2, adding $\Delta(S)$ to both sides and rearranging the terms, we get $\sum_i c_i(S_{-i}, S_i^*) \leq 2.5 \cdot cost(S^*) + \frac{\Delta(S)}{2}$. Therefore, $cost(S) \leq 2.5 \cdot cost(S^*) + \frac{3}{2}\Delta(S)$.

□

Next we show that unweighted congestion games with resources that satisfy the γ-bounded jump condition, satisfy the γ-bounded jump condition according to definition 2.2.

Definition 4.9. *(resource γ-bounded jump). Resource e satisfies the γ-bounded jump condition if its delay function satisfies $d_e(x+1) \leq \gamma \cdot d_e(x)$ for every $x \geq 1$, for $\gamma \geq 1$.*

Lemma 4.10. *Let Λ be a congestion game with nonnegative, non-decreasing delay functions in which every resource has γ-bounded jump. Then, the game Λ satisfies the γ-bounded jump condition according to definition 2.2.*

Proof. Consider any profile S and any player i with improving action S_i'. We first show the first property in definition 2.2. Consider any player j. Then,

$$c_j(S) - c_j(S_{-i}, S_i') \leq \sum_{e\in S_i\cap S_j} d_e(n_e(S)) \leq c_i(S).$$

For the second property in definition 2.2. Consider any player j with action S_j'. Then,

$$\begin{aligned}&c_j(S_{-\{i,j\}}, S_i', S_j') - c_j(S_{-j}, S_j')\\ &\leq \sum_{e\in(S_i'\setminus S_i)\cap S_j'} d_e(n_e(S_{-i}, S_i') + 1)\\ &\leq \sum_{e\in(S_i'\setminus S_i)\cap S_j'} \gamma\cdot d_e(n_e(S_{-i}, S_i'))\\ &\leq \gamma\cdot c_i(S_{-i}, S_i').\end{aligned}$$

Where the second inequality uses the assumption that each resource e has γ-bounded jump. □

Theorem 3.3, Lemmas 4.6, 4.10 and the fact that resource with linear latency function has 2-bounded jump, yield the following corollary.

Corollary 4.11. *Let $\frac{1}{8} > \delta \geq \alpha$. Consider a congestion game Λ with linear latency functions and any initial state S_{init}. The unrestricted α-Nash dynamics with liveness property generates a profile S with $cost(S) \leq 2.5(1 + O(\delta))OPT(\Lambda)$ in at most $O\left(\frac{n}{\alpha\delta}\log(\frac{\phi(S_{init})}{\phi^*})\cdot T\right)$ steps.*

4.2 Polynomial Latency Functions

In this section, we consider congestion games with polynomial latency functions of degree d. We show that congestion games with polynomial latency functions are β-*nice* according to definition 2.1 with $\beta = d^{d(1-o(1))}$. Price of anarchy results which appear in [9] imply that for $\beta = d^{d(1-o(1))}$ and for every profile S property (i) in definition 2.1 holds.

Lemma 4.12. *Congestion games with polynomial latency functions of degree d are β-*nice potential games with $\beta = d^{d(1-o(1))}$.

Theorem 3.3, Lemmas 4.12, 4.10 and the fact that resource with polynomial of degree d latency function has 2^d-bounded jump, yield the following corollary.

Corollary 4.13. *Let $\frac{1}{8} > \delta \geq \alpha$. Consider a congestion game Λ with polynomial latency functions of degree d and any initial state S_{init}. The unrestricted dynamics with liveness property generates a profile S with $cost(S) \leq d^{d(1-o(1))}(1 + O(\delta))OPT(\Lambda)$ in at most $O\left(\frac{2^d\cdot n}{\alpha\delta}\log(\frac{\phi(S_{init})}{\phi^*})\cdot T\right)$ steps.*

5. PROFIT MAXIMIZING CONGESTION GAMES

In this section, we extend the results for cost minimizing congestion games to profit maximizing congestion games. We first define some preliminaries for these games. Consider an exact potential game Λ. Let S be a profile of the players and let $S_i^{'}$ be a best response strategy for player i in strategy profile S. The payoff of player i in strategy profile S is denoted by $p_i(S)$ and each player wants to maximize its payoff. In this setting, for each player i, let $\Delta_i(S) = p_i(S_{-i}, S_i') - p_i(S)$ and let $\Delta(S) = \sum_i \Delta_i(S)$.

Definition 5.1. *An exact potential game Λ with potential function ϕ and social function $profit$ is β-*nice *iff for any state S it holds that*

1. $\beta\cdot(profit(S) + \Delta(S)) \geq OPT(\Lambda)$.
2. $\phi(S) \leq profit(S)$.

Definition 5.2. *(γ-Bounded Jump). Consider any profile S and any player i with improvement move S_i'. Then, for every player j the following properties hold:*

1. *for every player j, $p_j(S_{-i}, S_i') - p_j(S) \leq p_i(S_{-i}, S_i')$*
2. *for every improvement action S_j' of player j, it holds $p_j(S_{-j}, S_j') - p_j(S_{-\{i,j\}}, S_i', S_j') \leq \gamma\cdot p_i(S_{-i}, S_i')$*

5.1 Convergence of profit maximizing games

Similar to the proof of Theorem 3.3 for convergence of the unrestricted α-Nash dynamics in cost minimizing games, we can prove the following general theorem for convergence time of the α-Nash best-response dynamics in profit maximizing games.

Theorem 5.3. *Let $\frac{1}{8} > \delta \geq 4\alpha$. Consider an exact potential game Λ that satisfies the β-*nice *property and the γ-bounded jump condition. For any initial state S_{init} the unrestricted α-Nash best-response dynamics with liveness property generates a profile S with $\beta(1 + O(\delta))profit(S) \geq OPT(\Lambda)$ in at most $O\left(\frac{\gamma n}{\alpha\delta}\log(\frac{\phi^*}{\phi(S_{init})})\cdot T\right)$ steps.*

The proof of this theorem is very similar to that of Theorem 3.3 and is left to the full version.

6. CUT GAMES

In this section, we study convergence in cut games (also called the party affiliation games). We show that these games are *nice* games that satisfy the bounded jump condition. First, we show that cut games are 2-*nice* according to definition 5.1.

LEMMA 6.1. *Cut games are* β-nice *potential games with* $\beta = 2$.

PROOF. We need to show that for any strategy profile S, $2(profit(S) + \Delta(S)) \geq OPT$. To do so, we show that $2(profit(S) + \Delta(S)) \geq \sum_{v \in V(G)} w_v$. Given any strategy profile S, for any player v, either $p_v(S) > \frac{w_v}{2}$, or if $p_v(S) < \frac{w_v}{2}$, then $\Delta_v(S) \geq w_v - p_v(S) - p_v(S)$, thus $2(p_v(S) + \Delta(S)) \geq 2(w_v - p_v(S)) \geq 2(w_v - \frac{w_v}{2}) = w_v$. Therefore, the cut game is a 2-nice game. □

Next we show that cut games satisfy the 1-bounded jump condition.

LEMMA 6.2. *Cut games satisfy the* 1*-bounded jump property.*

PROOF. For two players u and v, if player u changes his strategy and goes to the same side as v, then payoff of v does not increase at all, thus $p_v(S_{-u}, S'_u) \leq p_v(S) + p_u(S_{-u}, S'_u)$. Otherwise, if player u changes his strategy to the other side of player v, the increase in the payoff of player v is at most $w_{u,v}$. Thus, $p_v(S_{-u}, S'_u) \leq p_v(S) + w_{u,v} \leq p_v(S) + p_u(S_{-u}, S'_u)$. This implies the first condition of the bounded jump property.

Now, consider a strategy profile S and two players u and v with two new strategies S'_u and S'_v. When player u changes his strategy to S'_u, if he decreases the payoff of strategy S'_v for player v, then it decreases this payoff by at most $w_{u,v}$. In this case, the payoff of u from switching to his strategy is at least $w_{u,v}$, therefore, $p_v(S_{-\{u,v\}}, S'_u, S'_v) \geq p_v(S_{-v}, S'_v) - p_u(S_{-u}, S'_u)$ which is the second condition of the bounded jump property.

□

Theorem 5.3 and Lemmas 6.1, 6.2 yield the following corollary.

COROLLARY 6.3. *Let* $\frac{1}{8} > \delta \geq 4\alpha$. *Consider a cut game* Λ *with and any initial state* S_{init}. *The unrestricted* α*-Nash best-response dynamics with a liveness property generates a profile* S *with profit at least* $\frac{1}{(2+O(\delta))} OPT(\Lambda)$ *in at most* $O\left(\frac{n}{\alpha\delta}\log(\frac{\phi(S_{init})}{\phi^*}) \cdot T\right)$ *steps.*

7. MARKET SHARING GAMES

In this section we consider market sharing games. We show that these games are 2-*nice* games that satisfy the 1-bounded jump condition. First, we show that market sharing games are 2-*nice* according to definition 5.1.

LEMMA 7.1. *Market sharing games are* β-nice *potential games with* $\beta = 2$.

PROOF. We need to show that for any strategy profile S, $2(profit(S) + \Delta) \geq OPT$. To do so, we can show that $profit(S) + \sum_{i \in N} p_i(S_{-i}, S'_i) \geq OPT$ where S'_i is the best response of player i in strategy profile S. Let S^* be the strategy profile of the optimal solution. Then $p_i(S_{-i}, S'_i) \geq p_i(S_{-i}, S^*_i)$. Let T be the set of markets that are satified in the optimal solution , i.e., $OPT = \sum_{j \in T} v_j$. Let R be the set of markets in T that are satisfied in S and L be the rest of markets in T. All of markets in R are satisfied in S, thus the sum of profits of markets in R is less than $profit(S)$. Moreover, for any market j in L, if $j \in S^*_i$, then the profit $p_i(S_{-i}, S^*_i)$ contains the whole value v_j of market j, since no other player plays this market. Therefore, $\sum_{j \in L} v_j \leq \sum_{i \in N} p_i(S_{-i}, S^*_i) \leq \sum_{i \in N} p_i(S_{-i}, S'_i)$. The above inequalities imply the 2-nice property as follows:

$$
\begin{aligned}
OPT &= \sum_{j \in T} v_j = \sum_{j \in R} v_j + \sum_{j \in L} v_j \\
&\leq profit(S) + \sum_{i \in N} p_i(S_{-i}, S'_i) \leq 2(profit(S) + \Delta).
\end{aligned}
$$

□

Next we show that market sharing games satisfy the 1-bounded jump condition.

LEMMA 7.2. *Market sharing games satisfy the* 1*-bounded jump property.*

PROOF. Consider two players i and i' in strategy profile S. If player i' changes his best response strategy to $S'_{i'}$, the congestion of each market j changes from vector n_j to n'_j where $n_j - 1 \leq n'_j \leq n_j + 1$. Then the increase in the payoff of player i is at most $\sum_{j \in S_i \cap (S_{i'} \backslash S'_{i'})} (\frac{v_j}{n_j - 1} - \frac{v_j}{n_j})$. The payoff of player i' after changing his strategy from S_i to $S'_{i'}$ is at least $\sum_{j \in S_{i'}} \frac{v_j}{n_j}$. For a market $j \in S_i \cap (S_{i'} \backslash S'_{i'})$, at least two players i and i' are playing market j in S, thus $n_j \geq 2$, thus $(\frac{v_j}{n_j - 1} - \frac{v_j}{n_j}) \leq \frac{v_j}{n_j}$. Therefore,

$$
\begin{aligned}
\sum_{j \in S_i \cap (S_{i'} \backslash S'_{i'})} (\frac{v_j}{n_j - 1} - \frac{v_j}{n_j}) &\leq \sum_{j \in S_i \cap (S_{i'} \backslash S'_{i'})} \frac{v_j}{n_j} \\
&\leq \sum_{j \in S_{i'}} \frac{v_j}{n_j} = p_{i'}(S).
\end{aligned}
$$

This implies the first condition of the bounded jump property, i.e, the increase in the payoff of player i is at most the payoff i'.

Consider a strategy profile S and two players i and i' with two best response strategies S'_i and $S'_{i'}$. When player i' changes his strategy to $S'_{i'}$, if he decreases the payoff of strategy S'_i for player i, then it decreases this payoff by at most $\sum_{j \in S'_i \cap (S'_{i'} \backslash S_{i'})} (\frac{v_j}{n_j} - \frac{v_j}{n_j + 1})$. In this case, the payoff of i' from switching to his strategy is at least $\sum_{j \in S'_{i'}} \frac{v_j}{n_j + 1}$. Since for any market $j \in S'_i \cap (S'_{i'} \backslash S_{i'})$, we have $n_j \geq 1$, thus, $\frac{v_j}{n_j + 1} \geq \frac{v_j}{n_j} - \frac{v_j}{n_j + 1}$. These inequalities imply the second condition of the 1-bounded jump property as follows:

$$
\begin{aligned}
& p_i(S_{-i}, S'_i) - p_i(S_{-\{i,i'\}}, S'_i, S'_{i'}) \\
& \quad \leq \sum_{j \in S'_i \cap (S'_{i'} \backslash S_{i'})} (\frac{v_j}{n_j} - \frac{v_j}{n_j + 1}) \\
& \quad \leq \sum_{j \in S'_{i'}} \frac{v_j}{n_j + 1} \leq p_{i'}(S_{-i'}, S'_{i'}).
\end{aligned}
$$

□

Theorem 5.3 and Lemmas 7.1, 7.2 yield the following corollary.

COROLLARY 7.3. *Let* $\frac{1}{8} > \delta \geq 4\alpha$. *Consider a market sharing game* Λ *with and any initial state* S_{init}. *The unrestricted* α*–Nash best-response dynamics with liveness property generates a profile* S *with profit* $\frac{1}{(2+O(\delta))}OPT(\Lambda)$ *in at most* $O\left(\frac{n}{\alpha\delta}\log(\frac{\phi^*}{\phi(S_{init})})\cdot T\right)$ *steps.*

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